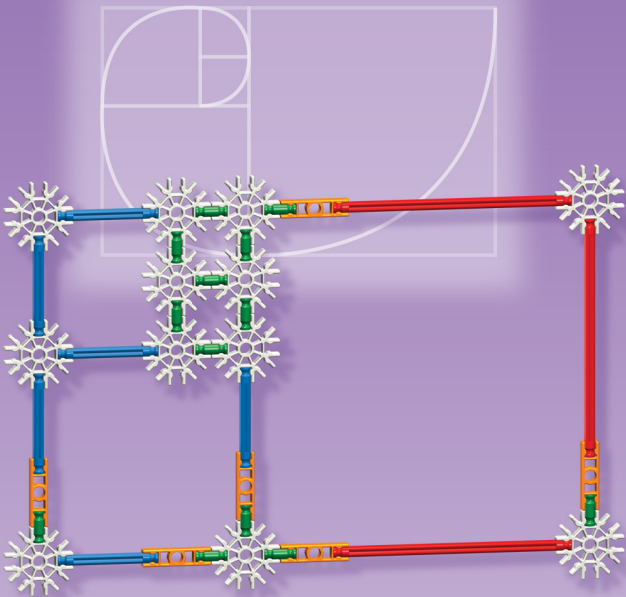
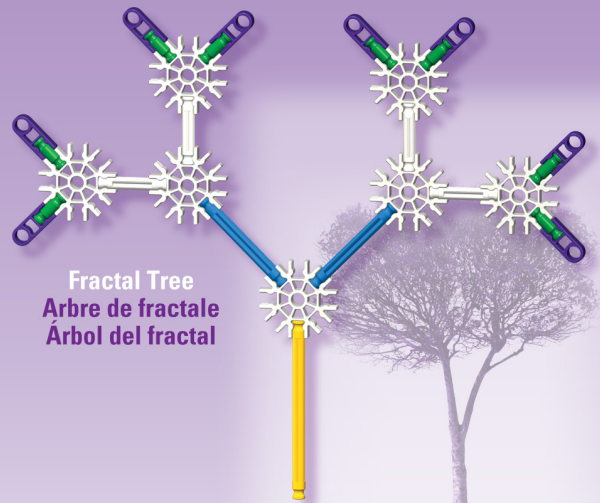


TEACHER'S GUIDE

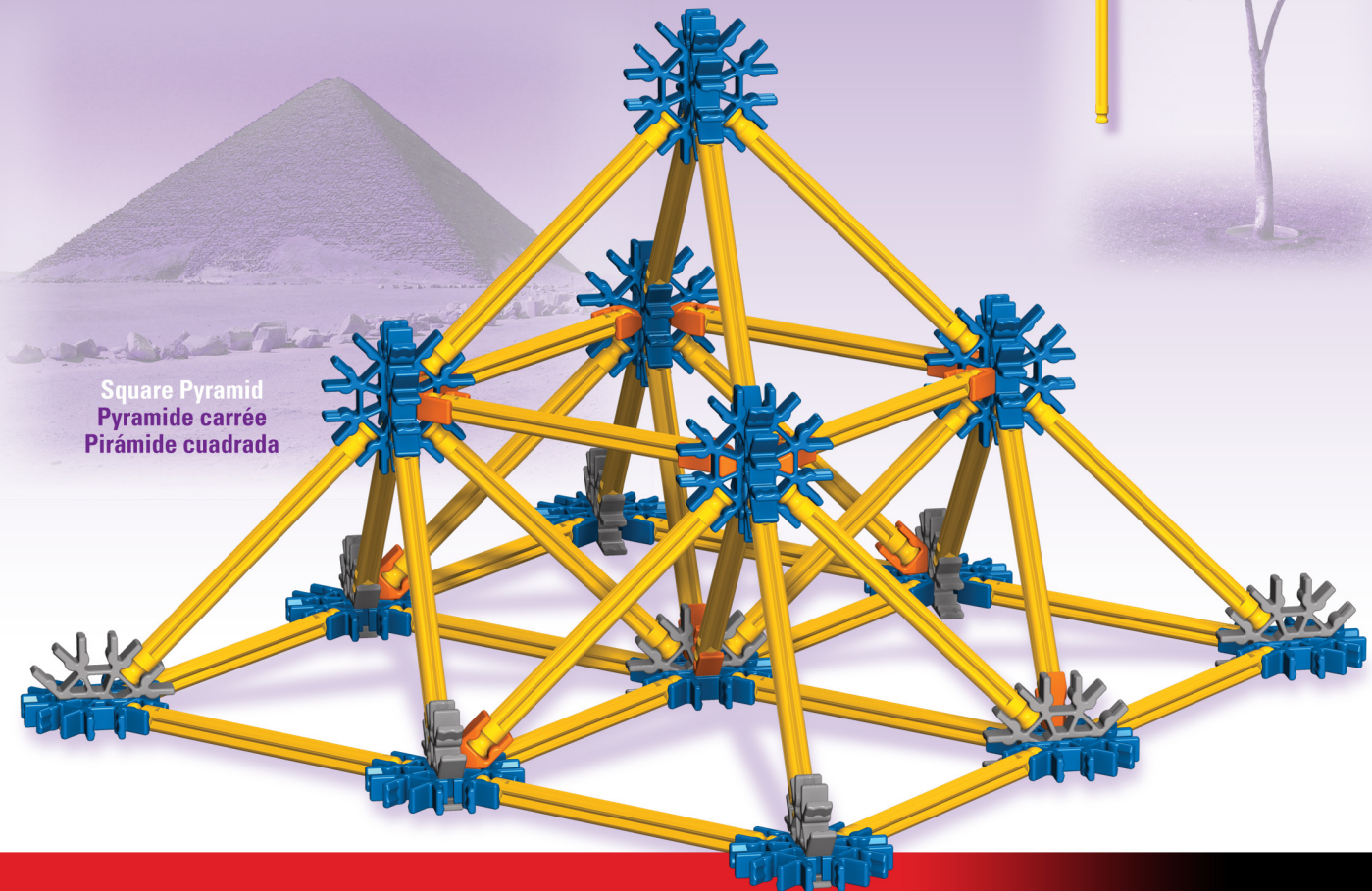
MIDDLE SCHOOL MATH™



Fibonacci Golden Spiral
Spirale d'or de Fibonacci
Espiral de oro de Fibonacci



Fractal Tree
Arbre de fractale
Árbol del fractal



Square Pyramid
Pyramide carrée
Pirámide cuadrada

MIDDLE SCHOOL MATH™

Teacher's Guide

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publication can be purchased from Research for
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Web site: www.knexeducation.com
Email: abcknex@knex.com
Phone: 1-888-ABC-KNEX (USA/CAN only)

K'NEX Limited Partnership Group
P.O. Box 700
Hatfield, PA 19440-0700
USA

Author - Dennis Ebersole:

- Professor, Mathematics & Computer Information
Science, Northampton Community College

Accomplishments:

- Author or co-author of six mathematics texts
- Principal Investigator/Project Director or co-PI for
various grants totally over \$2.5 million
- Section writer for *Beyond Crossroads*, the
American Mathematics Association of Two Year
Colleges mathematics standards; coeditor of
*IMAGES – Improving Measurement and Geometry
in Elementary Schools*
- Consultant for eastern PA school districts



WARNING:

CHOKING HAZARD - Small parts.
Not for children under 3 years.



AVERTISSEMENT:

DANGER D'ÉTOUFFEMENT - Pièces de petite taille.
Ne pas donner aux enfants de moins de 3 ans.

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$90^\circ < \angle A < 180^\circ$

$v = y$

$r + b = b + r$

CD

$y + s = s + y$

$y + y + w + w = 2y + 2w$

$90^\circ < \angle A < 180^\circ$

INTRODUCTION

Overview:

The Middle School Math Teacher's Guide was developed to provide the support you need to facilitate hands-on learning and the support your students need to investigate a wide variety of mathematics concepts that form the core of mathematics instruction in the middle grades.

In these times of high stakes testing and the national desire to make students competitive at the international level it becomes extremely important to have the very best instructional materials available for students to utilize as they explore the wonders of mathematics and use mathematics to model the world around them. As the educational community moves away from the rote memorization of the past and encourages teachers to offer students opportunities to explore the concepts behind the mathematics they are learning, the role, use and quality of manipulatives must change.

The K'NEX Education Middle School Math set provides exciting, dynamic materials and curriculum support to help you guide your students as they explore the patterns and relationships in their world and discover how middle school mathematical concepts can be used to understand and model those patterns and relationships.

K'NEX Middle School Math set:

This K'NEX mathematics set is designed to address critical mathematics concepts in the middle school classroom and to provide instructional models that will enhance students' understanding of important concepts and algorithms. The K'NEX construction system can trace its development to the world of mathematical relationships and thus shapes, structures, etc. made from K'NEX can be easily linked to mathematical concepts, as you will see in the 16 lessons in this set. With direction provided by this guide and the materials provided in this set teachers are able to offer students a program of study that uses hands-on exploration in conjunction with an inquiry-based approach to learning. As students work cooperatively they are encouraged to interact with each other as they build, investigate, discuss, and evaluate mathematical concepts, ideas, and models. The set also provides alternative assessment opportunities to better assess what all students know.

Teacher's Guide:

This guide is intended as a resource for teachers and students as they tackle meaningful mathematics content in the classroom. A series of comprehensive lessons include: student objectives, lesson overviews, materials lists, extension activities, and Student Inquiry sheets to provide teachers with all of the information they need to successfully work with their students. The Teacher Notes section at the beginning of this guide includes additional information that teachers will find to be invaluable as they seek to motivate students to explore, experience and learn mathematics. This additional information includes: an NCTM Standards alignment chart that includes reference to each lesson, assessment strategy ideas, suggested K'NEX mathematics conventions, and more.

Student Journals:

We suggest that students maintain a journal for the activities they complete from the K'NEX Middle School Math set. A loose-leaf format serves this purpose well as students complete Student Inquiry Sheets for each lesson. These sheets should be hole-punched and included along with notes, drawings, and conjectures made by the students to provide a comprehensive record of the growth of individual students. This information is an excellent source for assessment data.

TEACHER'S NOTES

The set that you received has enough materials to serve four groups of 3 – 4 students each working cooperatively on any of the activities designed for use with this kit in the classroom. The materials can also be divided into four smaller units and used to outfit a series of mathematics activity centers around the classroom. The compartmentalized tray that holds the K'NEX pieces may serve as a resource center for each of the groups to draw from during activities or you may choose to separate the materials into four smaller sets before the students begin to work.

As with any manipulative that is introduced to students, it is suggested that you provide time for students to explore the materials on their own at the beginning of their first session with the materials. It is human nature to explore and investigate so whether we provide time for that personal exploration or not, the students are going to find time to do it on their own. When you first introduce K'NEX in the classroom, ask for a show of hands to indicate which students have used K'NEX in other classrooms or at home. When you form groups for instruction, include an experienced K'NEX builder in each group.

Sections of the Teacher's Guide:

Teacher's Outline: the first page or two of each lesson provides topical information about each lesson.

- **Lesson length:** the average time or number of class periods you can expect to spend on the lesson.
- **Student Objectives:** list of information and goals you can expect your students to attain upon successful completion of the activity.
- **Overview of Lesson:** lists what the lesson will convey to students and how the lesson will flow.
- **Materials and Equipment:** lists the materials that will be required to complete the lesson activities.
- **Motivation and Introduction:** gives an introductory question or statement that will pique student interest.
- **Development:** the basic stages of the lesson.
- **Summary and Closure:** the types of information that students should provide to indicate the level of their understanding at the end of the lesson.
- **Assessment Activities:** suggestions for hands-on assessment of student comprehension of information.
- **Extensions:** listings of possible extension activities as appropriate.

National Council of Teachers of Mathematics Standards Alignments:

The K'NEX Education Middle School Math set for Grades 6 – 8 is designed for a standards-based environment.

- The lessons and activities in this Teacher's Guide were all crafted to provide students with an opportunity to not only meet but exceed the levels of mathematics understanding outlined in the National Council of Teachers of Mathematics Standards Alignments (NCTM) Standards.
- The NCTM Standards & Expectations chart, found on page 7 indicates which of the lessons address the various standards. This chart makes it possible for the

Student Inquiry Sheet: is a small group project that actively engages students in discovering the mathematics under consideration.

- You may make as many copies of the Student Inquiry Sheets as needed to support your program of instruction and number of students. It is important that you review these sheets in detail as you plan your lesson.
- These sheets include questions that students will answer and possibly turn in for an assessment when the lesson is completed. You may choose to require that students complete the sheets individually or that they submit a group sheet.
- Some lessons include challenge activities on the final Student Inquiry Sheet. These challenge activities provide additional assessment opportunities while allowing students to practice working with concepts they have explored and learned.

Lesson Answer Keys: download the Answer Keys, that coincide with the Student Response Sheets, from the web at www.knex.com/middle-school-math. These will help make the process of providing feedback to your students easier and more efficient.

teacher to select a group of lessons that align with a particular set of standards or to plan a series of lessons to support the local curriculum.

- K'NEX Education maintains and updates a section of the www.knexeducation.com web site that includes alignments of each of its mathematics sets with NCTM as well as state standards.
- Go to www.knexeducation.com, click on your state on the web site's standards map, then scroll down to the standards alignments for the Middle School Math set for Grades 6 – 8.

Mathematics Conventions with K'NEX:

The author of this Teacher's Guide and the staff at K'NEX Education have developed some mathematics conventions that will enable students to easily see relationships between their K'NEX creations and drawings, graphics, formulas, and descriptions found in their textbooks, workbooks, quizzes, or tests.

Line, Line Segment, and Ray:

- Refer to page 3 in the Instructions Booklet. On the left side of the page you will see models of a K'NEX line, line segment, and ray. These three models are made from red rods, white connectors, and red connectors.
- The three models can be made from any of the K'NEX rods (green, white, blue, yellow, red, or silver).
- Students can practice making a variety of these models using rods of various colors.
- The line segment on page 3 has two distinct endpoints which are highlighted with white connectors.
- Students can transfer the red line segment to paper by 1) placing the model on a sheet of paper, 2) using a crayon or pencil to place a dot in the center of each endpoint and 3) after removing the model, connecting the two endpoints with a ruler. The same process can be used to transfer ray and line to paper.
- In order to tell the three drawings apart once they are on paper, the two endpoints of the line segment and one endpoint of the ray should be made larger and darker. Arrow heads should be added to the drawings of the ray and the line where the red connectors had been. For naming purposes, two points on the line and a point on the ray just before the arrow heads should be darkened and enlarged.
- Any of the line segments can be identified based on its color. Lower case letters are assigned to serve as an abbreviation for each of the line segments and can be used to identify the line segment in written work or to identify the length of the line segment in formulas. The listing of line segments and their abbreviations can be found below. (If your curriculum or textbook places a bar above the abbreviations for line segments, please feel free to instruct students to do that when using these abbreviations.)

- Green line segment = g
- White line segment = w
- Blue line segment = b
- Yellow line segment = y
- Red line segment = r
- Silver line segment = s

Angles:

- Refer to page 3 in the K'NEX Instructions Booklet to see a variety of angles (two rays with a common endpoint) that can be formed using K'NEX pieces.
- The measures of fixed angles on standard K'NEX connector pieces are multiples of 45 degrees. Notice on page 3 that there are some angles that can be formed

with a two-piece hinge connector that allow you to form a wide range of angles. Any angle with measure from approximately 40° to 320° can be formed with the hinged connector pieces.

- These K'NEX angles can be transferred to paper.
- If the model is placed on a sheet of paper, a crayon, or pencil can be used to place a dot in the center of the common endpoint and each arrow head. When the model is removed, the dots can be connected with a ruler and arrow heads can be drawn on the ends of the rays. The angle can now be measured or used for other activities requiring pencil and paper. (The opening in a hinged angle is small. A sharp pencil or mechanical pencil may be required to transfer one of these angles to paper.)

Perimeter and Area Notation:

- To describe the perimeter of a K'NEX shape (see page 4 of the Instructions Booklet) students may use the abbreviations for the line segments that were introduced earlier. For example:
 - A square with red rods would have a perimeter of $4r$.
 - A rectangle with blue and white rods would have a perimeter of $2b + 2w$.
 - An octagon with yellow rods would have a perimeter of $8y$.
- To describe the area of a K'NEX shape students may use the abbreviations for the line segments that were introduced earlier and add a superscript 2 to indicate the square of a length. For example:
 - A square with red rods would have an area of r^2 .
 - A square with silver rods would have an area of s^2 .
 - A rectangle with yellow and white rods would have an area of $2w^2$ or $\frac{1}{2}y^2$, since the yellow segment is twice as long as the white segment.
- If students are given time to experiment with the concept of area using their K'NEX pieces they will identify many relationships that exist between the various abbreviations. They will also develop the ability to find the areas of complex polygons, either by decomposing the figure into simpler shapes or surrounding the shape with a rectangle and subtracting the areas of common shapes. Can the students find the perimeter and area of each polygon on page 14 of the Instructions Booklet?
- To describe the volume of a K'NEX shape (see page 18 of the Instructions Booklet) students may use the abbreviations for the line segments that were introduced earlier and add a superscript 3 to indicate the cube of a length. For example:
 - A cube with red rods would have a volume of r^3 .
 - A rectangular prism with four rectangular faces of blue and red line segments and two faces that are blue squares would have a volume of $2b^3$ or $\frac{1}{4}r^3$.
 - Can students find the volume of each polyhedron on pages 18 and 19 of the Instructions Booklet?



NCTM Standards & Expectations for Grades 6 – 8

Corresponding
Teacher's
Guide
Lesson #

NUMBER AND OPERATION

Understand numbers, ways of representing numbers, relationships among numbers and number systems:

- Work flexibly with fractions, decimals, and percents to solve problems 1, 2, 7, 8, 9, 16
- Understand and use ratios and proportions to represent quantitative relationships 1, 4, 7
- Develop an understanding of large numbers and recognize and appropriately use exponential, scientific, and calculator notation 14, 15
- Use factors, multiples, prime factorization, and relatively prime numbers to solve problems 6
- Develop meaning for integers and represent and compare quantities with them 10

Understand meanings of operations and how they relate to one another:

- Understand the meaning and effects of arithmetic operations with fractions, decimals, and integers 8, 10
- Use the associative and commutative properties of addition and multiplication and the distributive property of multiplication over addition to simplify computations with integers, fractions and decimals 10
- Understand and use the inverse relationships of addition and subtraction, multiplication and division, and squaring and finding square roots to simplify computations and solve problems 2, 8, 9, 10

Compute fluently and make reasonable estimates:

- Select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators, or computers, and paper and pencil, depending on the situation, and apply the selected methods 1, 2, 8, 9
- Develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use 8, 9, 10
- Develop, analyze, and explain methods for solving problems involving proportions, such as scaling and finding equivalent ratios 1, 4

ALGEBRA

Understand patterns, relations, and functions:

- Represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules 1, 4, 5, 7, 12, 13, 14, 15, 16
- Relate and compare different forms of representations for a relationship 12, 13, 15, 16
- Identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations 14, 15, 16

Represent and analyze mathematical situations and structures using algebraic symbols:

- Develop an initial conceptual understanding of different uses of variables 1, 4, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16
- Explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope 13
- Use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships 1, 4, 7, 12, 13, 15, 16



- Recognize and generate equivalent forms for simple algebraic expressions and solve linear equations . . . 13

Use mathematical models to represent and understand quantitative relationships:

- Model and solve contextualized problems using various representations, such as graphs, tables, and equations 1, 4, 13, 14, 15, 16

GEOMETRY

Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships:

- Precisely describe, classify, and understand relationships among types of two-dimensional and three-dimensional objects using their defining properties 3
- Understand relationships among the angles, side lengths, perimeters, areas, and volumes of similar objects 1, 2, 4
- Create and critique inductive and deductive arguments concerning geometric ideas and relationships, such as congruence, similarity, and the Pythagorean relationship 1, 2, 3, 4, 8, 9

Apply transformations and use symmetry to analyze mathematical situations:

- Examine the congruence, similarity, and line or rotational symmetry of objects using transformations 1, 4

Use visualization, spatial reasoning, and geometric modeling to solve problems:

- Use geometric models to represent and explain numerical and algebraic relationships 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16
- Recognize and apply geometric ideas and relationships in areas outside the mathematics classroom, such as, art, science, and everyday life 5

MEASUREMENT

Understand measurable attributes of objects and the units, systems and processes of measurement:

- Understand, select, and use units of appropriate size and type to measure angles, perimeter, area, surface area, and volume 1, 2, 4

Apply appropriate techniques, tools, and formulas to determine measurements:

- Select and apply techniques and tools to accurately find length, area, volume, and angle measures to appropriate levels of precision 15
- Solve problems involving scale factors, using ratio and proportion 1, 2, 4

PROCESS STANDARDS

Problem Solving:

- Build new mathematical knowledge through problem solving 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16

Reasoning and Proof:

- Make and investigate mathematical conjectures 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16

Communication:

- Organize and consolidate their mathematical thinking through communication 5
- Use the language of mathematics to express mathematical ideas precisely 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16



EF

E

F

$Y + S = S + Y$

$Y + Y + W + W = 2Y + 2W$

$W < r$

$90^\circ < \angle A < 180^\circ$



$90^\circ < \angle A < 180^\circ$

$v = y$

AB

A

B

$r + b = b + r$

C

D

CD

Connections:

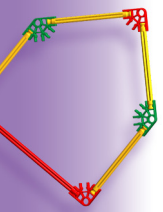
- Recognize and use connections among mathematical ideas 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16

Representation:

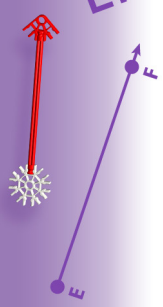
- Create and use representations to organize, record, and communicate mathematical ideas 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16
- Use representations to model and interpret physical, social and mathematical phenomena 5, 14

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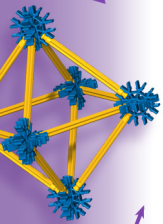
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$$y + s = s + y$$

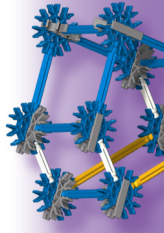
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$$w < r$$



$90^\circ < \angle A < 180^\circ$

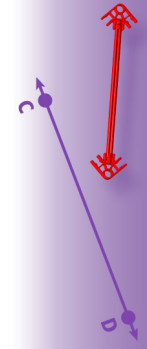
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$y = y$
 \overline{AB}



$$r + b = b + r$$



\vec{CD}



LESSON 1

Similar Shapes

Lesson Topics: Similarity, dilations, ratios, and proportions

Lesson Length: Two 50-minute periods

Student Objectives:

Students will:

- Understand how a dilation of a 2-D shape affects the perimeter and area of the shape.
- Use proportions to find the perimeter or area of a similar shape, given information about the original shape.
- Use the Pythagorean Theorem to find the length of the hypotenuse of a right triangle.
- Use inductive reasoning to make predictions.

Grouping for Instruction:

- Whole group for the launch and closure sessions.
- Small groups of 3 – 4 students each will complete the hands-on activities.

Overview of Lesson:

- Students will use K'NEX pieces to create first similar rectangles and later similar isosceles triangles.
- They will compare the perimeters of the similar shapes and use inductive reasoning to describe the pattern they discover.
- They will compare the areas of the similar shapes and describe the pattern relating the dilation factor to the areas.
- They will use proportions to determine an unknown length, perimeter, or area of a similar shape.

Materials and Equipment:

- *K'NEX rods per group:*
 - 4 white
 - 4 yellow
 - 4 silver
 - 10 green
 - 12 blue
 - 22 red
- *K'NEX connectors per group:*
 - 6 light gray
 - 16 red
- *Calculator*
- *Copies of the Lesson #1 Student Inquiry Sheets*

A – Motivation and Introduction:

1. "If the length and width of the computer screen are both increased by a factor of 1.2, how much has the area of the screen increased? If the length of a side of a square pizza is increased by a factor of 1.5, how much more should you be willing to pay for the pizza? In this investigation you will learn the answers to these and similar questions."
2. Instruct the class that they will use K'NEX pieces to create similar rectangles and triangles as they explore the relationship between a change in the dimensions of the shapes and a change in their perimeter and area.

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Ask students to predict what will happen to the circumference and area of a square if you double the length of the radius. Ask students what they would expect to happen to the surface area and volume of a rectangular solid, if you double the length of each edge.

Assessment:

Observe the students during the group work. Use a checklist to record whether students are using correct terminology and reasoning mathematically. Each group should receive a group grade on the activity. Ask the students to explain what they learned during the lesson in their Math Journal. Students should also record any concepts or ideas that are not clear to them at this time.

Extensions:

Similar Isosceles Right Triangles

Do the patterns discovered during the investigations work for all similar polygons? Repeat the experiment completed with rectangles during the lesson but this time use isosceles right triangles (such as one made from K'NEX with blue rods for legs and a yellow rod for the hypotenuse.) If the same patterns hold, we will feel more comfortable saying these patterns hold for all similar polygons. Show how you determined whether the patterns hold.



Similar Shapes: Ratios and Proportions

To make the polygons in this activity we need to create sides with integer values. We will say a line segment formed by a green rod and two connectors has a length of 1. Then a line segment formed using one blue rod and two connectors has a length of 2.

Now create a segment using one red rod and two connectors.

1. What is the length of this segment, if the green segment has a length of 1?

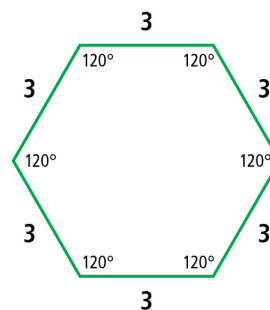
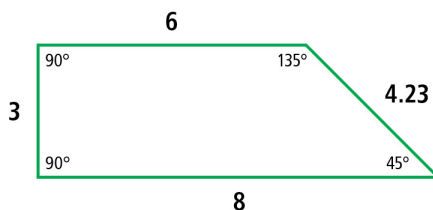
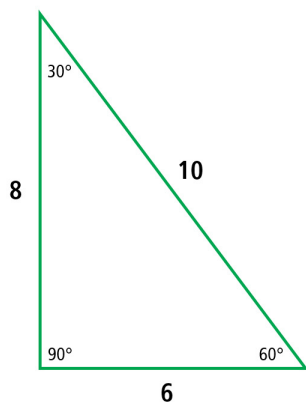
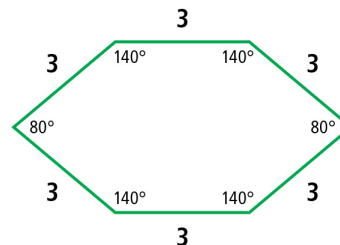
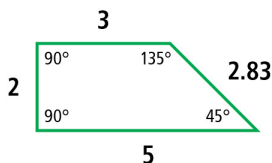
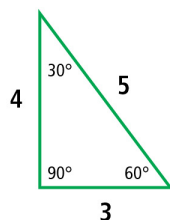
2. How did you find the length of the red rod segment?

Using these three segments we can make a segment of any desired length. For example, if you combine a red segment and a green segment you obtain a segment with length 5.

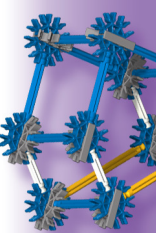
3. How could you create a segment of length 7?

Two polygons are similar, if 1) the corresponding interior angles have the same measure and 2) the ratios of corresponding sides are equal to a constant.

4. Which of the pairs of polygons below are similar polygons? Circle any pair that is similar. Justify your answers.



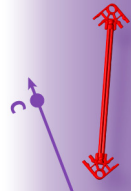
$90^\circ < \angle A < 180^\circ$



$v = y$



$r + b = b + r$



$w < r$



$90^\circ < \angle A < 180^\circ$

LESSON 1

Similar Shapes: Rectangles

Create a 1 x 2 rectangle.

1. What is the **perimeter** of this rectangle? (How many green segments would be needed to surround the rectangle?)

$$P_1 =$$

2. What is the **area** of this rectangle? (How many 1 x 1 squares are needed to fill in the rectangle?)

$$A_1 =$$

Create a 2 x 4 rectangle.

3. Is this rectangle similar to the 1 x 2 rectangle? Justify your answer using the definition of similar polygons (see Glossary).

4. What is the **perimeter** of this rectangle?

$$P_2 =$$

5. What is the **area** of this rectangle?

$$A_2 =$$

6. What is the ratio of any two corresponding sides for these two rectangles?

Dilation factor:

(We call this ratio the **dilation factor**, because we can think of the second rectangle as a dilation of the first rectangle.)

Create a rectangle similar to the 1 x 2 rectangle with a dilation factor of 3.

7. What are the **dimensions** of this new rectangle?

8. What is the **perimeter** of this rectangle?

$$P_3 =$$

9. What is the **area** of this rectangle?

$$A_3 =$$

Create a rectangle similar to the 1 x 2 rectangle with a dilation factor of 4.

10. What are the **dimensions** of this new rectangle?

$$P_4 =$$

11. What is the **perimeter** of this rectangle?

12. What is the **area** of this rectangle?

$$A_4 =$$

Create a 2 x 3 rectangle.

13. What is the **perimeter** of this rectangle?

$$P_1 =$$

14. What is the **area** of this rectangle?

$$A_1 =$$

Create a rectangle similar to the 2 x 3 rectangle with a dilation factor of 2.

15. What are the **dimensions** of this new rectangle?

16. What is the **perimeter** of this rectangle?

$$P_2 =$$

17. What is the **area** of this rectangle?

$$A_2 =$$

Create a rectangle similar to the 2 x 3 rectangle with a dilation factor of 3.

18. What are the **dimensions** of this new rectangle?

19. What is the **perimeter** of this rectangle?

$$P_3 =$$

20. What is the **area** of this rectangle?

$$A_3 =$$



Create a rectangle similar to the 2x3 rectangle with a dilation factor of 4.

21. What are the dimensions of this new rectangle?

22. What is the perimeter of this rectangle?

$P_4 =$

23. What is the area of this rectangle?

$A_4 =$

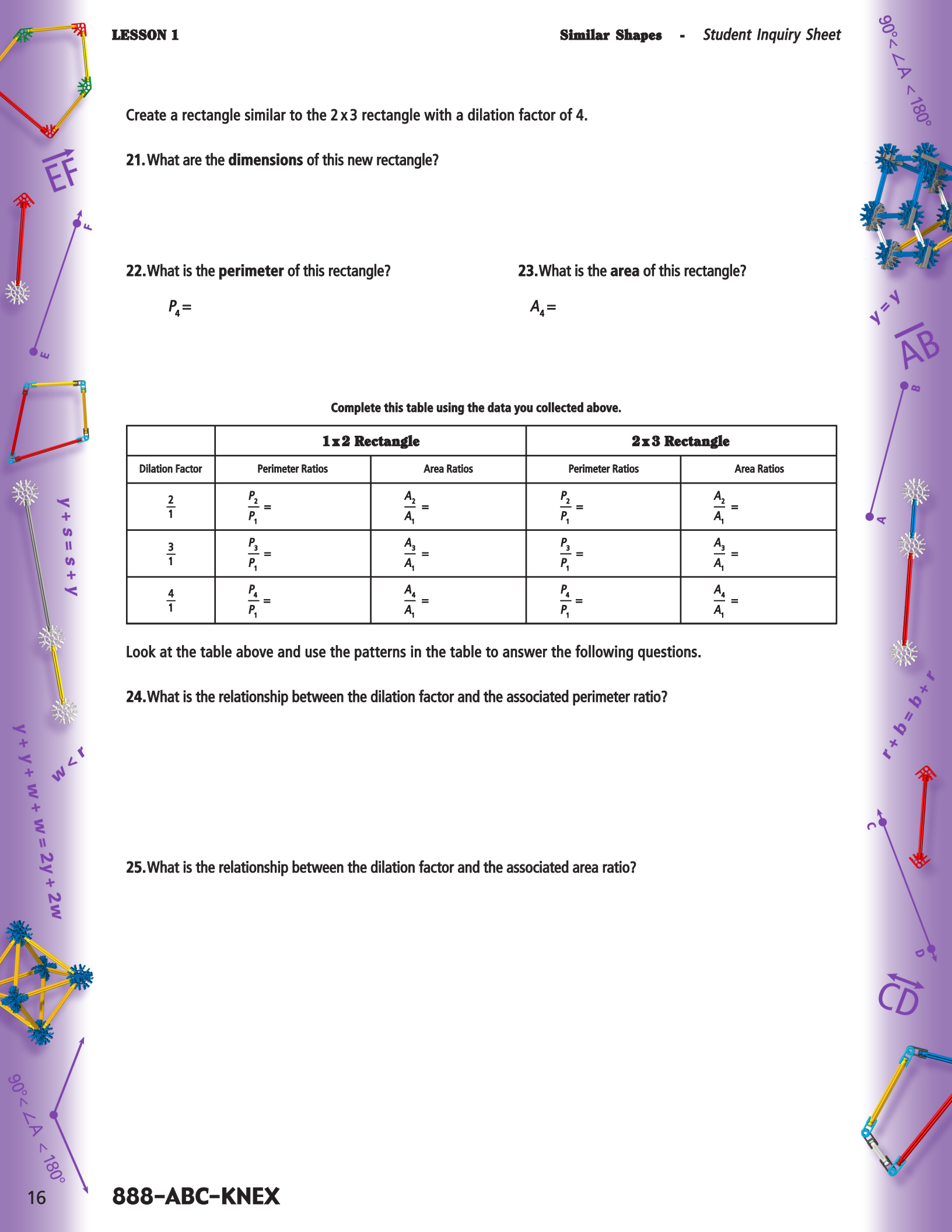
Complete this table using the data you collected above.

Dilation Factor	1 x 2 Rectangle		2 x 3 Rectangle	
	Perimeter Ratios	Area Ratios	Perimeter Ratios	Area Ratios
$\frac{2}{1}$	$\frac{P_2}{P_1} =$	$\frac{A_2}{A_1} =$	$\frac{P_2}{P_1} =$	$\frac{A_2}{A_1} =$
$\frac{3}{1}$	$\frac{P_3}{P_1} =$	$\frac{A_3}{A_1} =$	$\frac{P_3}{P_1} =$	$\frac{A_3}{A_1} =$
$\frac{4}{1}$	$\frac{P_4}{P_1} =$	$\frac{A_4}{A_1} =$	$\frac{P_4}{P_1} =$	$\frac{A_4}{A_1} =$

Look at the table above and use the patterns in the table to answer the following questions.

24. What is the relationship between the dilation factor and the associated perimeter ratio?

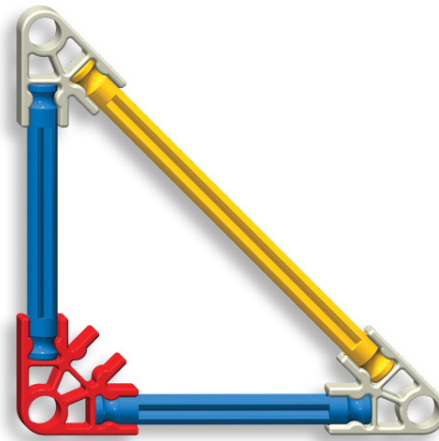
25. What is the relationship between the dilation factor and the associated area ratio?



**Extension Activity:
Similar Isosceles Right Triangles**

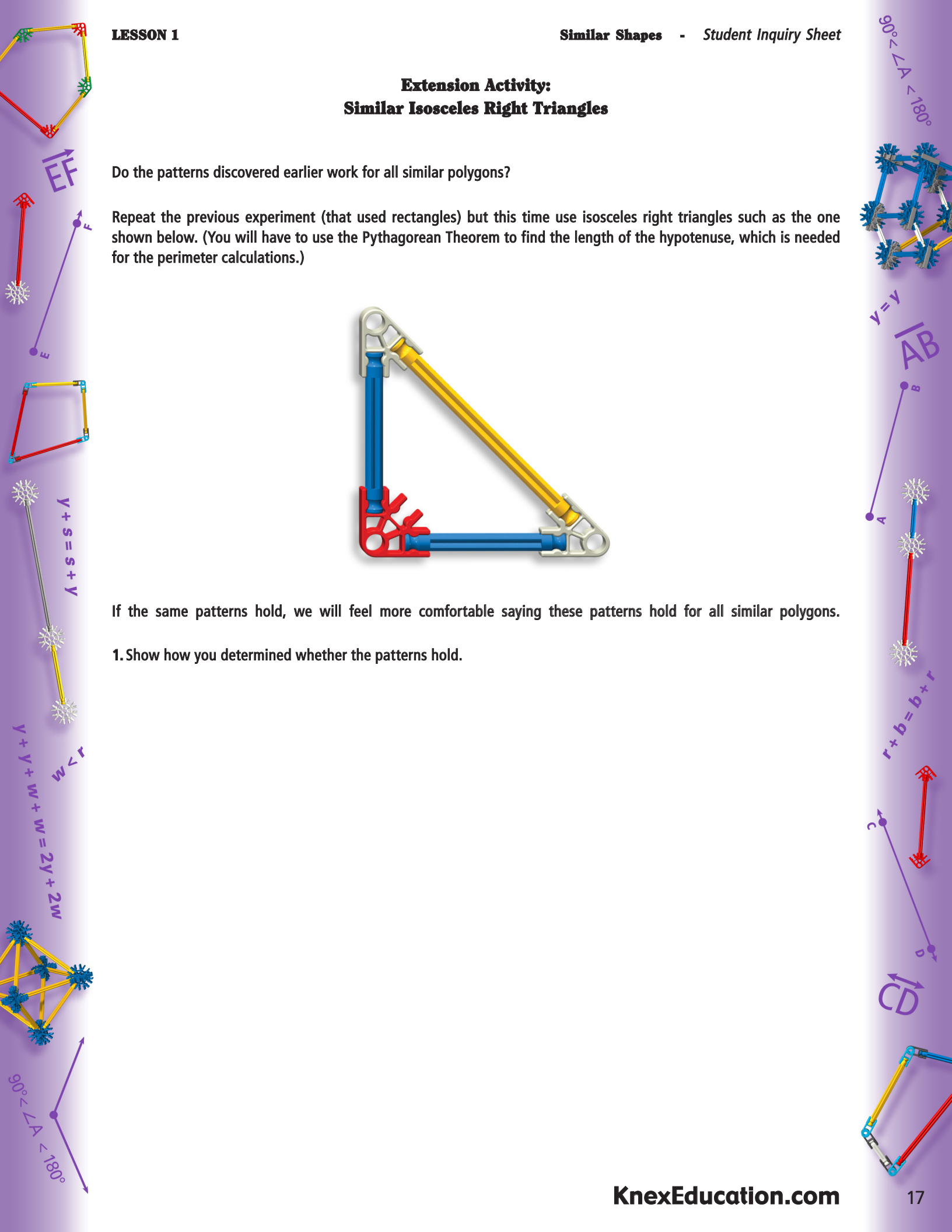
Do the patterns discovered earlier work for all similar polygons?

Repeat the previous experiment (that used rectangles) but this time use isosceles right triangles such as the one shown below. (You will have to use the Pythagorean Theorem to find the length of the hypotenuse, which is needed for the perimeter calculations.)



If the same patterns hold, we will feel more comfortable saying these patterns hold for all similar polygons.

1. Show how you determined whether the patterns hold.



LESSON 2

Nesting Squares

Lesson Topics: Similarity, midpoints, area, fractions, and fractals.

Lesson Length: Two 50-minute periods

Student Objectives:

Students will:

- Understand that a midpoint divides a line segment into two equal segments.
- Use geometric reasoning to find the area of an inscribed square.
- Use the square root as the inverse of squaring (in $A = x^2$) to find the length of the side of a square.
- Use inductive reasoning to make predictions.
- Understand self-similarity.

Grouping for Instruction:

- Whole group for launch and closure
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX to create nested squares where the area of each new square will be half the area of the previous square.
- They will use midpoints as the vertices of the new squares.
- They will find the area of the new square using geometric reasoning.
- They will multiply fractions to find the area of squares created after the first nested square.
- They will use the Pythagorean Theorem to find the exact length of a side of a square.
- They will be introduced to the concepts of self-similarity and fractals.

Materials and Equipment:

► *K'NEX rods per group:*

- 6 red
- 6 silver
- 10 green
- 12 white
- 12 blue
- 12 yellow

► *K'NEX connectors per group:*

- 8 red
- 11 yellow
- 12 white

► *Copies of the Lesson #2 Student Inquiry Sheets*

A – Motivation and Introduction:

1. "If you start with a square and create a quadrilateral inside the square so its vertices are on the square and divide the sides in half, what will the area of this new shape be? During this activity you will discover answers to this and other questions."
2. Inform the class that they will use K'NEX pieces to create similar, inscribed quadrilaterals and explore the areas and side lengths of the new quadrilaterals.

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

In this activity we started with a square. Suppose we had started with a rectangle instead. How would this change your results? What kind of shape would be formed at each stage? What could you say about the relationship between the areas of the original rectangle and the inscribed rectangle?

Assessment:

Observe the students during group work. Use a checklist to record whether students are using correct terminology and reasoning mathematically. Each group should receive a group grade on the project. Ask the students to explain what they learned during the lesson and any concepts that are still unclear to them in their Math Journal.

Extensions:

Repeat this investigation but start with a right triangle. The resulting model represents a Sierpinski's Triangle which is a well-known fractal. A later lesson in this guide will address Sierpinski's Triangles in detail.



Nesting Squares: Fractals and Similar Shapes

Create a square (figure 1) using 4 silver K'NEX rods and 4 red connectors. Let the length of a side of this square equal 1 unit.

1. What is the area of the square in terms of this unit length? Explain or show how you found the area of the square in the space below.

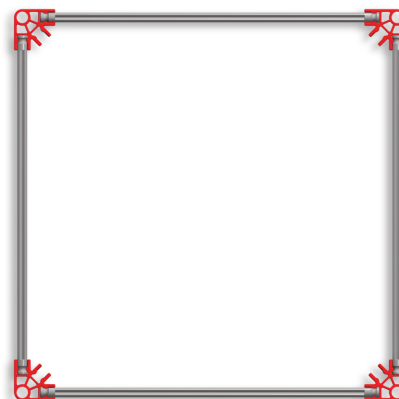


Figure 1

Create a square (figure 2) that is congruent with your silver square which uses two rods of equal length for each of its sides. Use a red connector for each of the vertices of the new square. Use white connectors to connect each of the two rods that make up a side of your new square.

2. What is the term used to describe the point that is at the halfway point of a line segment?

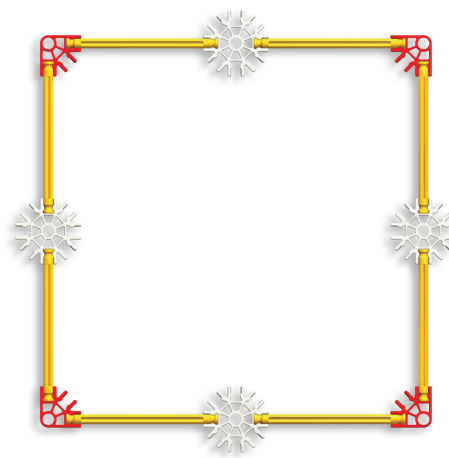


Figure 2

3. If the length of a side of the original square is 1 unit, what is the length of the segment with label x in the diagram above?

Create a quadrilateral (four-sided polygon) inside the new square that was just created (figure 3).

4. What kind of shape is the quadrilateral just formed? How do you know?

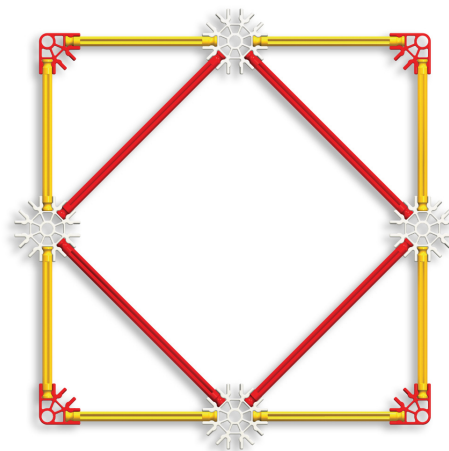


Figure 3

5. What is the area of the new quadrilateral, if the area of the original square is 1 square unit? Describe how you found your answer.

6. Use the Pythagorean Theorem to find the length of a side of the inscribed square now that you know its area.

EF

$y + s = s + y$

$y + y + w + w = 2y + 2w$

$w < r$

$90^\circ < \angle A < 180^\circ$

$90^\circ < \angle A < 180^\circ$

$v = y$

AB

A B

$r + b = b + r$

$90^\circ < \angle A < 180^\circ$

LESSON 2

Replace the center square with a congruent square using two blue rods per side (figure 4). Connect the midpoints of this square to create a new quadrilateral inside the one just formed (figure 5).

7. What kind of shape is the quadrilateral just formed? How do you know?

8. What is the length of each side of this quadrilateral? How did you find the answer?

9. What is the area of this new quadrilateral? How did you find the answer?

Repeat this process to create another quadrilateral inside this one. (For the remainder of this activity, use yellow connectors for all vertices and mid points.)

10. What type of shape have you formed?

10. If the area of the original square was 1 square unit, what is the area of the new quadrilateral? How did you find the area?

11. What is the length of a side of this quadrilateral?

Repeat this process again to complete the final shape.

12. What type of shape did you form?

13. What is the area of this new quadrilateral?

14. What is the length of each side of this quadrilateral?

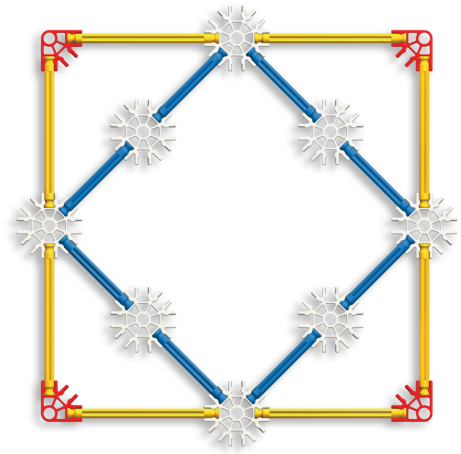


Figure 4

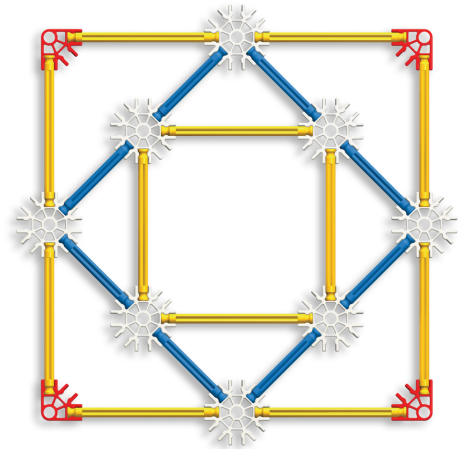
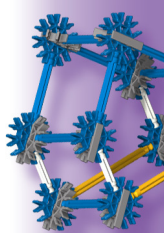


Figure 5

The shape you have formed is a **fractal**, a geometric shape that is self-similar. That is, if you look at part of the shape it will be similar to the whole shape.

15. Describe how your final shape is self-similar.

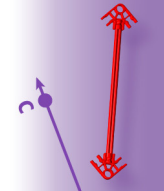
$90^\circ < \angle A < 180^\circ$



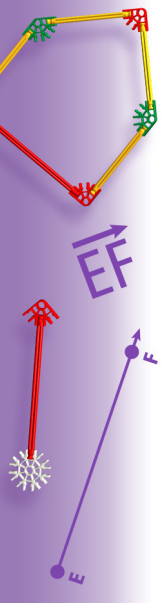
$v = y$



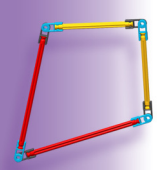
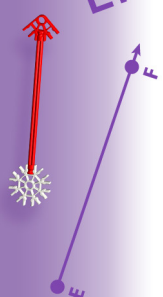
$r + b = b + r$



CD



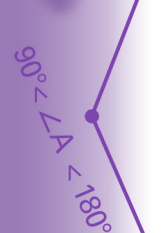
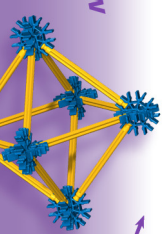
EF



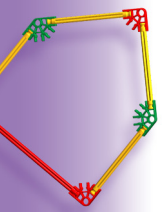
$Y + S = S + Y$



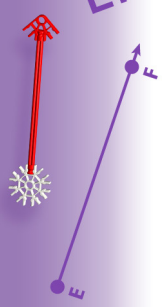
$Y + Y + W + W = 2Y + 2W$



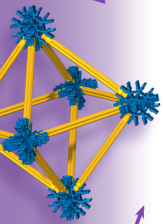
$90^\circ < \angle A < 180^\circ$



\vec{EF}

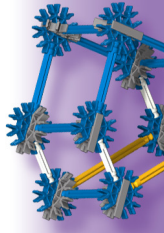


$$Y + Y + W + W = 2Y + 2W$$

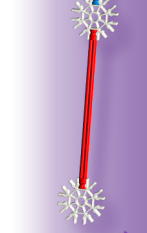


$90^\circ < \angle A < 180^\circ$

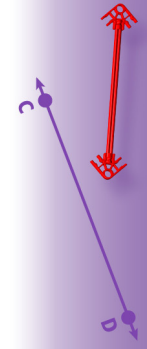
$90^\circ < \angle A < 180^\circ$



$v = v$
 \overline{AB}



$$r + b = b + r$$



\vec{CD}



LESSON 3

Polyhedra and Euler's Formula

Lesson Topics: Prisms, pyramids, Euler's formula

Lesson Length: Three 50-minute periods

Student Objectives:

Students will:

- Know and use the definitions of the terms polyhedron, vertex, edge, face, prism, pyramid, parallelogram, convex, and concave.
- Discover Euler's formula using a pattern they identify and analyze in a table.
- Be able to name different prisms and pyramids.
- Know which Platonic solid is a prism and which is a pyramid.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups of 3 – 4 students each for the investigation.

Overview of Lesson:

- Students will use K'NEX to create different prisms and pyramids.
- They will use the number of corner connectors as the number of vertices, and the number of rod segments as the number of edges.
- By creating tables showing the number of vertices V , faces F , and edges E of different prisms and pyramids, they will discover Euler's Formula: $E = V + F - 2$.
- They will determine that the cube and equilateral triangular pyramid are Platonic solids.

Materials and Equipment:

- K'NEX Middle School Math set and Instructions Booklets
- K'NEX rods per group:
 - 10 green
 - 12 white
 - 12 silver
 - 16 blue
 - 20 yellow
 - 20 red
- K'NEX connectors per group:
 - 6 orange (small 45°)
 - 16 dark gray
 - 16 blue
- Ask one group to save a copy of the pyramid model from page 9 of the Instructions Booklet. Remove the top 2 connectors and the top 4 yellow rods to form a truncated pyramid to use during the Closure of the lesson.
- Copies of the Lesson #3 Student Inquiry Sheets

A – Motivation and Introduction:

1. "Can you give me an example of a prism? Can you give me an example of a pyramid?"
2. Inform the class that they will use K'NEX to create prisms and pyramids and discover a formula that holds true for all such surfaces.

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project. Some students may have difficulty building right triangular prisms and oblique prisms, instruct them to follow page 5 of the Instructions Booklet. Some may have difficulty building triangular pyramids, instruct them to pages 6, 7, 10, 24, and 25 of the Instructions Booklet. These models are triangular pyramids.
4. Each group will report what they learned from the lesson to the entire class

C – Summary and Closure:

Show the class a truncated pyramid. Ask them to discuss in their groups whether it is a prism, pyramid, or neither. Have them go back to the definitions to discover that it cannot be a prism for two reasons: the sides are not parallelograms and the two bases are not congruent. Also, it is not a pyramid because the sides are not triangles and the sides do not meet at a point (the apex).

Assessment:

Observe the students during the group work and use a checklist to record whether students are using correct terminology and reasoning mathematically. Each group should receive a group grade on the project. Ask students to explain in their Math Journal what they learned during the lesson and any concepts that are still unclear.

Extensions:

Ask students to see if they can create any other Platonic solids. They will need the small orange 3-D connectors to create an equilateral triangular pyramid. Directions are included on pages 24 and 25 of the Instructions Booklet if students need assistance with the equilateral pyramid.



Polyhedra and Euler's Formula

Simple closed surfaces:

A simple closed surface is a surface that divides space into three distinct regions: 1) points in the interior of the surface, 2) points on the surface, and 3) points in the exterior of the surface. A sphere is an example of a simple closed surface. In this activity we will investigate properties of a subset of simple closed surfaces called polyhedra (plural of polyhedron). A polyhedron is a simple closed curve composed of polygons joined on their edges. A sphere is not a polyhedron, but a cube is a polyhedron.

We can form some common polyhedra using K'NEX. You can place two blue connectors, two dark gray connectors, or one blue and one dark gray connector together to create the connectors required for the polyhedron that are shown on pages 5 and 10 of the Instructions Booklet. You will also use the special 3-D orange connector for some constructions. That connector is also demonstrated in the Instructions Booklet on page 8.

Create a yellow cube using 12 yellow rods, eight dark gray connectors and eight blue connectors.

A face of a polyhedron is one of the polygons that make up the polyhedron. A face must lie in a single plane. A vertex of a polyhedron is a point where three or more faces meet. In a K'NEX representation, a vertex is a double connector (blue and gray). An edge is a line segment where two faces meet. In a K'NEX representation, an edge is a rod.

Prisms:

A prism is a polyhedron with the same cross-section throughout. A prism must have a base and top that are congruent polygons and lateral faces that are parallelograms.

1. Is the cube you created a prism? Justify your answer.

2. How many faces does your cube have?

Faces $F =$

3. How many vertices (plural of vertex) does the cube have?

Vertices $V =$

4. How many edges does the cube have?

Edges $E =$

A **dihedral angle** is a 3-D angle. While angles in 2-D are formed by two rays that have a common vertex, a dihedral angle is formed by two planes that meet in a line. Any two faces of a polyhedron that share an edge form a dihedral angle.

5. What is the measure of every interior dihedral angle for the cube?

The cube is an example of a **right prism** – a prism where the dihedral angles formed by the base and each side parallelogram has a measure of 90°. Right prisms are named according to the type of polygon used to form their base.

6. Create right prisms with the following bases. Find the number of faces, vertices, and edges for each prism. Fill in the table below. Refer to page 5 of the Instructions Booklet to find a model of a prism with a right triangle base.

Oblique Prism Base	# of Vertices V	# of Faces F	# of Edges E
Square			
Right Triangle			
Rectangle			
Rhombus (not a square)			
Isosceles Trapezoid			
Pentagon			



Euler's Formula:

The famous 18th century mathematician Leonhard Euler (pronounced "oiler") discovered a relationship between the number of faces and vertices of a prism and the number of edges of the prism.

7. Can you see the pattern in the table (on the previous page)? Explain.

8. Can you write a formula involving F , V , and E that can be used to find the number of edges of a prism? Show your formula.

All of the prisms you have created so far have had rectangles for the side faces.

9. Will Euler's Formula hold true if the side faces are parallelograms that are not rectangles?

These are called oblique prisms. With oblique prisms the top and bottom bases are not aligned one directly over the other.

10. Create oblique prisms with the following bases, fill in the table below, and determine if Euler's Formula holds true for oblique prisms. (Page 5 of the Instructions Booklet includes instructions for an oblique hexagonal prism that you can use for the fourth row of the chart below.)

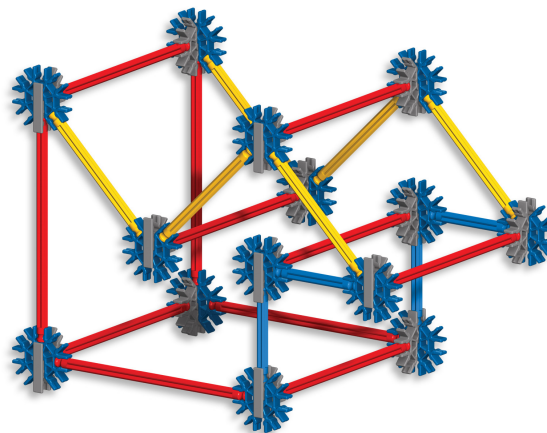
Oblique Prism Base	# of Vertices V	# of Faces F	# of Edges E
Square			
Right Triangle			
Rectangle			
Hexagon			
Isosceles Trapezoid			
Pentagon			

11. Does Euler's Formula hold true for these prisms?

12. Do you think it holds for all prisms? Explain why?

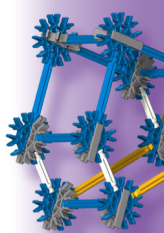
Most likely all of the prisms you have made are convex polyhedra. This means that if you choose any two interior points of the polyhedron, the line segment connecting the two points must lie completely inside the polyhedron.

If you can find two interior points of a polyhedron where the line segment connecting the two points lies partly outside the polyhedron, we say the polyhedron is a concave polyhedron. Below is a concave prism where the two bases are congruent concave polygons.



13. Do you think Euler's Formula will hold true for concave prisms? Explain why?

$90^\circ < \angle A < 180^\circ$



$v = v$
 \overline{AB}



$r + b = b + r$



$r + b = b + r$



\overline{CD}



14. Test your conjecture by using K'NEX to create several concave prisms, finding the number of faces, vertices, and edges of each prism, and filling in the table below. Use the table to determine if each concave prism satisfies Euler's Formula. Use the name of the base to name each prism. For example, concave pentagon. (Refer to pages 19, 20, and 21 of the Instructions Booklet for additional concave polyhedra.)

Concave Prism	# of Vertices <i>V</i>	# of Faces <i>F</i>	# of Edges <i>E</i>

15. Does Euler's Formula hold true for concave prisms? How do you know?

Pyramids:

A pyramid consists of a base that is a polygon and lateral sides that are all triangles that meet in a common point, called the apex. Pyramids are named based on the polygon used as their base. For example, a square pyramid has a square for its base.

16. Do you think Euler's Formula will hold true for pyramids? Explain why.

17. Create as many different pyramids as you can. Record the name, number of faces, number of vertices, and number of edges of each one you made. (See pages 6-7, 8-9, 10, and 24-25 of the Instructions Booklet for directions to build a variety of pyramids.)

Pyramid	# of Vertices <i>V</i>	# of Faces <i>F</i>	# of Edges <i>E</i>
Triangular Pyramid with Equilateral Faces			
Triangular Pyramid with 3 Right Triangle Faces			
Square Pyramid			

18. Does Euler's Formula hold true for these pyramids? Explain why.

Platonic Solids

A Platonic solid is a regular polyhedron. This means that the closed surface is convex, the faces are all congruent regular polygons, and every interior dihedral angle is congruent. The last condition is difficult to check. However, the condition is satisfied if the same number of faces meet at each vertex, so we use this condition instead. That is, for a polyhedron to be a Platonic solid, it must be a closed, convex polyhedron composed of congruent, regular polygons with the same number of polygons meeting at each vertex.

19. Which prism is a Platonic solid?

18. Which pyramid is a Platonic solid?



LESSON 4

Dilations of 3-D Shapes

Lesson Topics: Similarity, dilations, surface area, volume, ratios, and proportions

Lesson Length: Two 50-minute periods

Student Objectives:

Students will:

- Use correct terminology when describing different 3-D shapes and their properties.
- Understand how a dilation of a 3-D shape affects the surface area and volume of the shape.
- Use proportions to find the surface area or volume of a similar 3-D shape, given information about the original shape.
- Use inductive reasoning to make predictions.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX to create similar cubes and prisms.
- They will compare the surface areas of the similar shapes and use inductive reasoning to describe the pattern.
- They will compare the volumes of the similar shapes and describe the pattern relating the dilation factor to the volumes.
- They will use proportions to determine an unknown surface area or volume of a similar shape.

Materials and Equipment:

- *K'NEX rods per group:*
 - 16 blue
 - 16 green
 - 20 red
- *K'NEX connectors per group:*
 - 10 orange (straight)
 - 11 yellow
 - 16 dark gray
 - 20 blue
- *Calculator*
- *Copies of the Lesson #4 Student Inquiry Sheets*

A – Motivation and Introduction:

1. "With landfills rapidly filling up manufacturers are concerned about packaging, because of the amount of cardboard or other packaging materials that will have to be discarded by consumers. This is directly related to the surface area of the package. Manufacturers are also concerned about the size of packaging, because if it is too large they may give the consumer more than promised or the consumer may complain because it looks like the box is only partially filled. How do errors in size affect the amount of a commodity such as candy or nails that will fit in the container? This is related to the volume of the container. In this investigation you will learn how changes in the dimensions of a 3-D shape affect the surface area and volume of the shape."
2. Inform the class that they will use K'NEX to create similar 3-D shapes and explore the relationship between a dilation of a shape, its surface area and volume.

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

1. Ask students to predict what will happen to the surface area and volume of a pyramid if you double both the base dimensions and the height of the pyramid.
2. Ask students if they think the same relationships will hold for cylinders. Require students to justify their conjectures.

Assessment:

Observe the students during the group work. Use a checklist to record whether students are using correct terminology and reasoning mathematically. Each group should receive a group grade on the project. Ask the students to explain in their Math Journal what they learned during the lesson and any concepts that are still unclear.

Extensions:

Ask students to use K'NEX to demonstrate that if you double the edge lengths of a regular tetrahedron, the surface area of the larger tetrahedron will be 4 times the surface area of the original tetrahedron. If students experience trouble, refer them to the tetrahedron models on pages 6-7 and 10 of the Instructions Booklet.



Dilations of 3-D Shapes: Relating Surface Area and Volume

Construction Notes: To simplify computations and to make the 3-D shapes in this activity we need to create edges with integer values. We will let a line segment formed by a green rod and two connectors have length 1. Then a line segment formed using a blue rod and two connectors will have length 2. Create a segment using a red rod and two connectors.

1. What is the length of this segment if the green segment has length 1?

2. How did you determine the length of the red segment?

Using these three segments we can make a segment of any desired length. For example, if you combine a red segment and a green segment you obtain a segment with length 5.

3. How could you create a segment of length 7?

We will use blue and dark gray connectors as the corners for the cubes and prisms constructed during this activity.

A **polyhedron** is a closed surface composed of polygons that each lie in a different plane with the property that the polygons meet at line segments called **edges** with only two polygons meeting at each edge. The corners of a polyhedron are called **vertices**. The polygons that form the surface are called **faces** of the polyhedron. Most polyhedra (plural of polyhedron) are named based on the number of faces they have. For example, an octahedron has 8 faces (Refer to page 10 in the Instructions Booklet).

When two faces meet at an edge their planes form a 3-D angle (called a **dihedral angle**). Two polyhedra are **similar**, if:

- A. they have the same number of faces,
- B. the corresponding faces are similar polygons,
- C. the ratios of corresponding edges are all the same, and
- D. the corresponding dihedral angles are congruent.

The last condition is difficult to check, so we often require that the same number of polygons meet at corresponding vertices. This, along with the other conditions, guarantees that the dihedral angles will have the same measures.

In order to understand surface area and volume we start with a unit cube. Use 12 green rods and 16 blue or dark gray connectors to create a unit cube – a cube with all edges of length 1.

Look at one of the faces of this cube. It is a square.

What is the area of this square? It is one square unit.

This will be the basic unit of area that you will use throughout this lesson.



LESSON 4

The **surface area** of a polyhedron is the sum of the areas of the faces of the polyhedron.

4. What is the surface area of the unit cube?

$$S_1 =$$

The amount of space inside this cube is 1 cubic unit. This will be our basic unit of **volume** (the amount of space inside a closed 3-D shape).

$$V_1 = 1$$

Now dilate the basic cube by a factor of 2. This means that the length of each edge of your new cube will be two times the length of an edge in the original cube. Create the cube using 12 blue rods and 16 blue or dark gray connectors.

5. Look at a face of this cube. How many unit squares would be needed to cover the area of one face?

6. What is the surface area of the new cube that has been dilated by a factor of 2? (The subscript of 2 indicates the edges have a length of 2.)

$$S_2 =$$

7. What is the ratio of the two surface areas?

$$\frac{S_2}{S_1} =$$

8. Look at this dilated cube again. If the unit cube had points with no dimension for vertices instead of the blue connectors, how many of these unit cubes would fit inside the new cube? This is the volume of this cube.

$$V_2 =$$

9. What is the ratio of the two volumes?

$$\frac{V_2}{V_1} =$$

Dilate the cube with an edge length of 2 by a factor of 2. The new cube will have an edge length of 4.

10. Look at a face of this cube. How many unit squares are needed to cover the area of one face?

Dilations of 3-D Shapes - Student Inquiry Sheet

11. What is the surface area of the entire cube with an edge length of 4 units?

$$S_4 =$$

12. What is the ratio of the surface areas of the cube with an edge length of 4 and the cube with an edge length of 2?

$$\frac{S_4}{S_2} =$$

13. Look at this dilated cube again. If the unit cube had points with no dimension, how many of the unit cubes would fit inside the new cube? This is the volume of the cube with an edge length of 4 units.

$$V_4 =$$

14. Find the ratio of the volumes of the cubes with edge length 4 and edge length 2?

$$\frac{V_4}{V_2} =$$

15. The cube with edge length 4 is a dilation of factor 4 of the unit square. Find the ratio of the surface areas of these two cubes.

$$\frac{S_4}{S_1} =$$

16. Find the ratio of the volumes of these two cubes.

$$\frac{V_4}{V_1} =$$

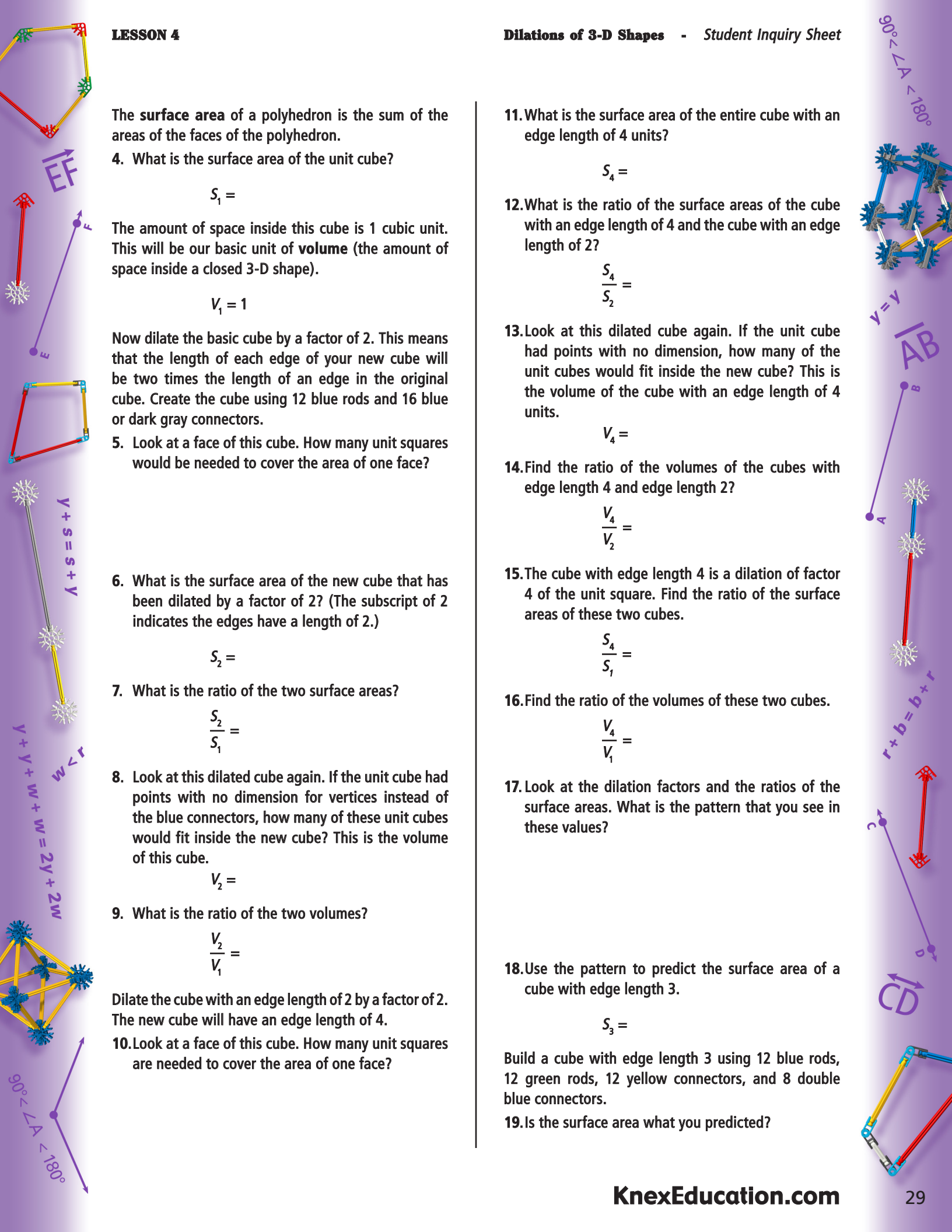
17. Look at the dilation factors and the ratios of the surface areas. What is the pattern that you see in these values?

18. Use the pattern to predict the surface area of a cube with edge length 3.

$$S_3 =$$

Build a cube with edge length 3 using 12 blue rods, 12 green rods, 12 yellow connectors, and 8 double blue connectors.

19. Is the surface area what you predicted?



20. Look at the dilation factors and the ratios of the volumes. What is the pattern?

21. Use the pattern to predict the volume of a cube with edge length 3.

$$V_3 =$$

22. Look at the cube with edge length 3. Is the volume what you predicted?

Prisms:

Create a rectangular prism with a base of one unit square and a height of 2 units. Find the surface area and volume of the prism (prism #1).

$$S_1 = \quad V_1 =$$

Create a prism that is a dilation of prism #1 by a factor of 2.

23. Find the surface area and volume of the second prism (prism #2) with a dilation factor of 2.

$$S_2 = \quad V_2 =$$

24. Find the ratio of the surface areas of these two prisms. Does the pattern you found with the ratio of surface areas for cubes still hold?

$$\frac{S_2}{S_1} =$$

25. Find the ratio of the volumes of these two prisms. Does the pattern you found with the ratio of volumes for cubes still hold?

$$\frac{V_2}{V_1} =$$

Create a prism that is a dilation of this prism by a factor of 3.

26. Find the surface area and volume of the prism (prism #3) with a dilation factor of 3.

$$S_3 = \quad V_3 =$$

27. Find the ratio of the surface areas of prism #3 and prism #1. Does the pattern you found with the ratio of the surface areas for cubes still hold true?

$$\frac{S_3}{S_1} =$$

28. Find the ratio of the volumes of prism #3 and prism #1. Does the pattern you found with the ratio of volumes for cubes still hold true?

$$\frac{V_3}{V_1} =$$

29. If you dilated the original prism by a factor of 4, what would be the surface area and volume of the new prism? How did you find the answers?

$$S_4 = \quad V_4 =$$

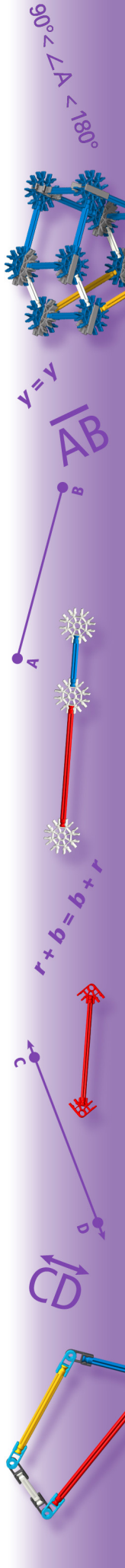
Create the prism and check to see if you were correct.

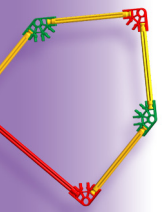
30. If you dilated the original prism by a factor of $\frac{1}{2}$, what would be the surface area and volume of the new prism? How did you find the answers?

$$S = \quad V =$$

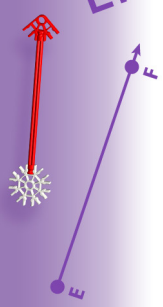
31. Do you think these relationships would also hold for pyramids? Why or why not?

32. For other polyhedra? Why or why not?

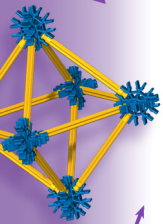




\vec{EF}



$$Y + Y + W + W = 2Y + 2W$$



$90^\circ < \angle A < 180^\circ$

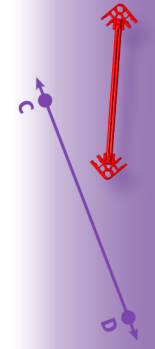
$90^\circ < \angle A < 180^\circ$



$v = v$
 \overline{AB}



$$r + b = b + r$$



\overline{CD}



LESSON 5

Pascal's Triangle

Lesson Topics: Patterns and fractal figures.

Lesson Length: One 50-minute periods

Student Objectives:

Students will:

- Find patterns in Pascal's triangle and use the patterns to extend the triangle.
- Understand modulo 2 and use it to form a mod 2 version of Pascal's triangle.
- Find patterns in the mod 2 version of Pascal's triangle.
- Recognize a fractal figure.
- Find patterns in a fractal figure.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation

Overview of Lesson:

- Students will start with the first few lines of Pascal's triangle and look for patterns in the triangle.
- They will then use the patterns to extend the triangle. Modulo 2 will then be used to convert the numbers in Pascal's triangle to either 0 or 1 depending on their modulo 2 representation. (Even numbers become 0 and odd numbers become 1.) This new triangle will then be represented by a triangle.
- Students will use K'NEX to create a triangle that models the mod 2 Pascal's triangle.
- They will note that this triangle is a fractal figure and find patterns in the triangles of their shapes.

Materials and Equipment:

- **K'NEX rods:**
24 red
36 yellow
- **K'NEX connectors:**
16 white
20 blue
11 yellow
- *Copies of the Lesson #5 Student Inquiry Sheets*

A – Motivation and Introduction:

1. "Blaise Pascal was a 17th century mathematician. In order to be more successful at games of chance, he used a triangle of numbers that shows the likelihood of certain events. Because he made the triangle famous, it now bears his name. In this lesson you will become familiar with Pascal's triangle and explore a version of the triangle that results in a fractal figure."
2. The teacher will define fractal figure and give examples of fractals in nature such as clouds and broccoli.

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
 - During the final phases of the activity, the students will construct a Pascal Triangle model that will be quite impressive if all four groups pool their pieces and make the largest triangle possible.

3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Do you think you will get another fractal figure if you convert the numbers in Pascal's triangle to their modulo 3 equivalents? Why?

Assessment:

Observe the students during the group work. Use a checklist to record whether students are working cooperatively and reasoning mathematically. Each group should receive a group grade on the project. Ask the students to explain in their Math Journal what they learned during the lesson and any concepts that are still unclear.

Extensions:

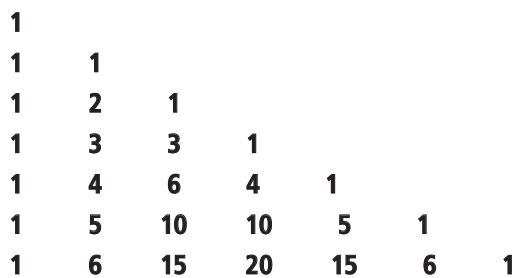
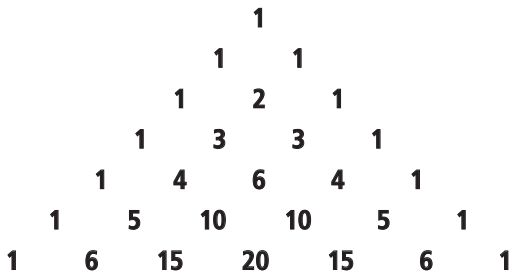
Ask students to do an online search of "Pascal's triangle and fractals" and find examples of other Pascal's triangles and fractals.



Pascal's Triangle and Fractals

Pascal's triangle:

The triangle that bears the name Pascal's triangle was known long before Blaise Pascal used it to better understand the probability of certain events. Lines 0, 1, 2, ... 6 are shown below for both an equilateral version and a right triangle version of Pascal's triangle. For the purposes of this lesson we will use the right triangle version.



1. Describe as many patterns as you can in Pascal's triangle.

2. Use the patterns to find the next two lines of the triangle. Draw those two lines below.

An interesting pattern develops in Pascal's triangle if we replace the numbers in the triangle by the remainder when each number is divided by 2. This remainder is called the value of the number modulo 2. These values for the first few counting numbers are shown below. The abbreviation "mod" is used for modulo.

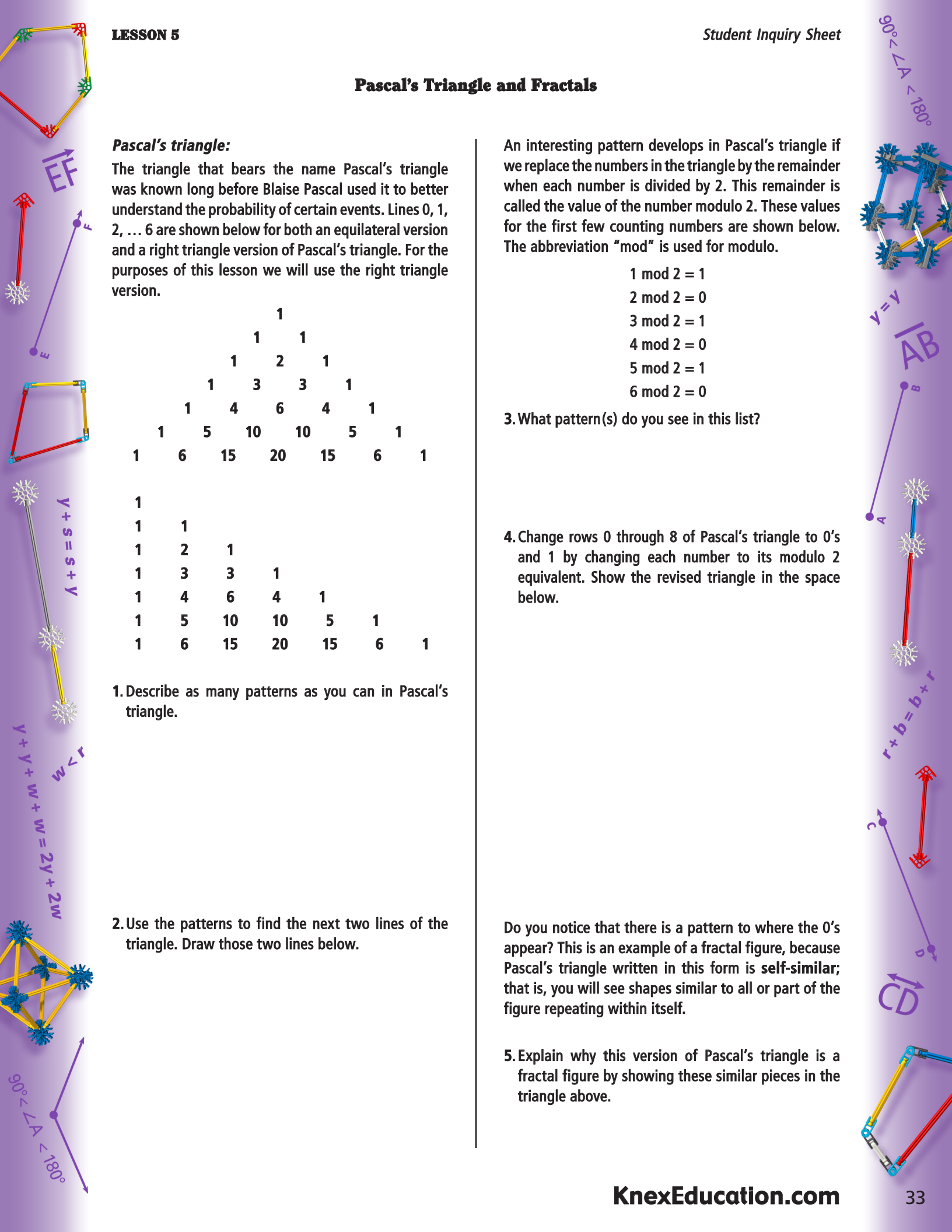
- 1 mod 2 = 1
- 2 mod 2 = 0
- 3 mod 2 = 1
- 4 mod 2 = 0
- 5 mod 2 = 1
- 6 mod 2 = 0

3. What pattern(s) do you see in this list?

4. Change rows 0 through 8 of Pascal's triangle to 0's and 1 by changing each number to its modulo 2 equivalent. Show the revised triangle in the space below.

Do you notice that there is a pattern to where the 0's appear? This is an example of a fractal figure, because Pascal's triangle written in this form is **self-similar**; that is, you will see shapes similar to all or part of the figure repeating within itself.

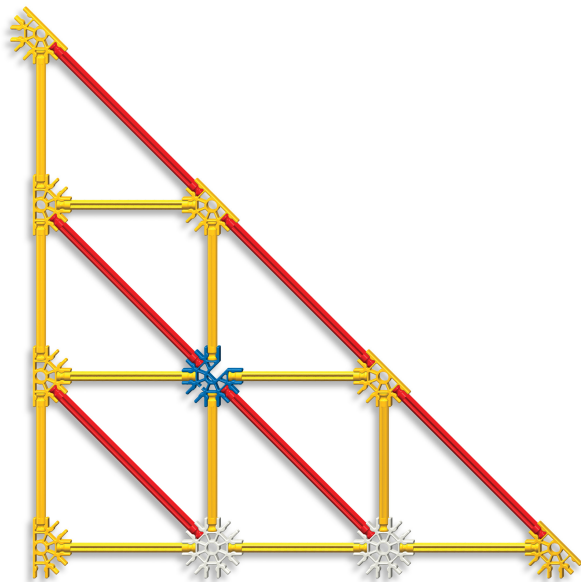
5. Explain why this version of Pascal's triangle is a fractal figure by showing these similar pieces in the triangle above.



6. Now we will create a triangle using K'NEX rods and connectors that better illustrates this self-similarity in a visual manner through varied size and color. The rules for creating the triangle are as follows.

- The numbers in the mod 2 triangle will all be represented by K'NEX connectors.
- The 1 digits in the mod 2 triangle will be represented by white or yellow connectors.
- The 0 digits in the mod 2 triangle will be represented by a blue connector.
- All the vertical and horizontal segments will be yellow rods.

(Refer to the model on the right side of page 23 in the Instructions Booklet. Construct the triangle but replace the yellow connectors in the center of the bottom row with a blue connector to meet the rules stated above. The diagram below shows the first three rows of the mod 2 triangle that you have constructed plus one additional row that you can add at this time.)



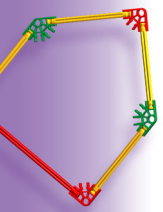
Although our triangle is not equilateral, it will show the same patterns as we would find if we had in fact created an equilateral triangle.

7. Create the triangle for as many lines of the mod 2 triangle as you can with the K'NEX available. Be sure to follow the rules as you build. Groups should collaborate to create the biggest Pascal's Triangle possible.

8. Look at the final triangle. What patterns do you see with the blue connectors?

9. What other patterns do you see in the triangle? Describe as many as you can.





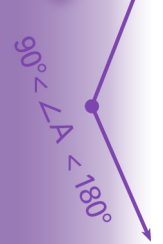
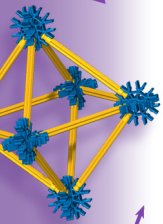
EF



$$y + s = s + y$$

$$y + y + w + w = 2y + 2w$$

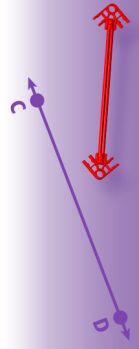
$$w < r$$



$$90^\circ < \angle A < 180^\circ$$



$$r + b = b + r$$



CD



LESSON 6

Rectangles and Factors

Lesson Topics: Factors, composite and prime numbers, deficient, abundant and perfect numbers.

Lesson Length: Two 50-minute periods

Student Objectives:

Students will:

- Use rectangles to find the factors of a counting number.
- Recognize whether a counting number is prime, composite, or neither.
- Be able to determine if a number is deficient, abundant, or perfect.
- Recognize square numbers.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX to create all possible rectangles with a specified area – the target counting number.
- They will interpret the dimensions of the rectangles as factors of the target number. Based on whether there is 1, 2, or more factors, they will determine whether the target number is the unit, a prime, or a composite number.
- They will also use the sum of the proper factors to determine if the target number is deficient, abundant, or perfect.
- They will discover that squares are the only numbers with an odd number of factors.

Materials and Equipment:

- *K'NEX rods per group:*
24 red
16 blue
12 green
- *K'NEX connectors per group:*
11 yellow
16 white
20 red
- *Rulers.*
- *Copies of the Lesson #6 Student Inquiry Sheets*

A – Motivation and Introduction:

1. "Banks transfer funds all over the world many times each day. Their customers do not want others to be able to access their private information as these transactions take place. When you buy online, using a credit card, you do not want others to have access to your number. One of the most common methods of encrypting data and messages sent over the Internet uses large prime numbers. In fact, you can help in finding new primes. GIMPS, the Great Internet Mersenne Prime Search, uses unused computer processor time on thousands of computers to search for larger and larger primes."
2. Inform the class that they will use K'NEX to create rectangles and explore factors of counting numbers.

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.

3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Use polling and A, B, C, D cards to determine if students learned the material. For example, one question might be as shown below.

- The factors of 40 are 1, 2, 4, 5, 8, 10, 20, and 40. Thus, 40 is?

A. deficient B. abundant C. perfect D. prime

Assessment:

Observe the students during the group work. Use a checklist to record whether students are using correct terminology and reasoning mathematically. Each group should receive a group grade on the project. Ask the students to explain in their Math Journal what they learned during the lesson and any concepts that are still unclear.

Extension:

Assign students to research Mersenne primes and to write a report about them.



LESSON 6

A counting number c is said to be a **composite number**, if more than one rectangle with counting number dimensions can be formed that have an area of c .

6. What must be true about the number of factors of a composite number?

7. Explain your reasoning.

8. Is the counting number 15 prime or composite? Use K'NEX rectangles to justify your answer.

9. What are the factors of 15?

A **proper factor** of a counting number c is a factor of c less than c . The number 9 has two proper factors: 1 and 3.

10. What are the proper factors of 15?

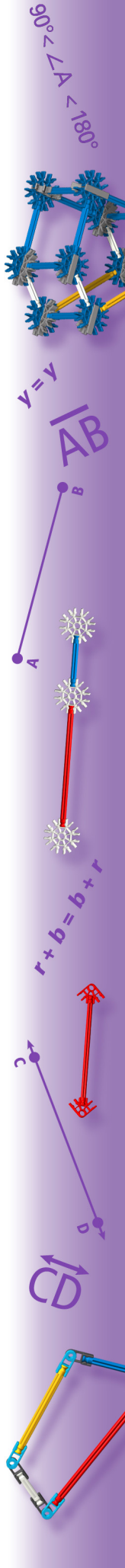
We say a counting number c is **deficient**, if the sum of its proper factors is less than c . Thus, 9 is deficient, since $1 + 3 = 4 < 9$.

11. Find another deficient number. Use rectangles to justify your answer.

A counting number c is **abundant**, if the sum of its proper factors is more than c .

Finally, a counting number c is **perfect**, if the sum of its proper factors equals c .

12. Is 15 deficient, abundant, or perfect? Use rectangles in the justification of your answer.



LESSON 6

12. Is 6 deficient, abundant, or perfect? Use rectangles in the justification of your answer.

A counting number c is called a square, if you can create a square with counting number dimensions that has area c square units.

16. Find the first five square counting numbers. Show the squares that you used to find each square number.

13. Is 12 deficient, abundant, or perfect? Use rectangles in the justification of your answer.

17. Find the factors of each of these square numbers. (Note that the first square, 1, has only 1 factor.) How many factors does each of the other squares have?

18. If a counting number c has an odd number of factors, what do you know about the number c ?

19. Find the second perfect counting number. Use rectangles in your proof that it is a perfect number.

Use one of the words "deficient", "abundant", or "perfect" to complete the following sentence:

14. All prime numbers are _____.

15. Explain your reasoning.



LESSON 7

Fibonacci Numbers

Lesson Topics: Fibonacci sequence, patterns, the golden mean, and the Fibonacci spiral.

Lesson Length: Two 50-minute periods

Student Objectives:

Students will:

- Use rectangles formed from successive squares to find terms of the Fibonacci sequence.
- Use patterns to create a recursive formula for finding the next term of the sequence.
- Use patterns to create a recursive formula for finding the next term of the Fibonacci fractions sequence.
- Create the Fibonacci spiral.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX to create rectangles with dimensions equal to successive pairs of terms in the Fibonacci sequence of numbers. The next rectangle will be formed by adding a square to the long side of the current rectangle.
- Students will use the dimensions of the resulting rectangles to find terms of the Fibonacci sequence.
- Then they will use the terms of the Fibonacci sequence to find Fibonacci fractions – fractions formed by finding the ratio of consecutive Fibonacci numbers.
- They will then find decimal approximations for these fractions and discover that the decimals approach a special number called the golden mean.
- Finally, students will create the Fibonacci spiral.

Materials and Equipment:

- **K'NEX rods:**
24 red
24 blue
16 green
- **K'NEX connectors per group:**
16 white
10 straight orange
- **Rulers**
- **Copies of the Lesson #7 Student Inquiry Sheets**

A – Motivation and Introduction:

1. "The ancient Greeks believed that certain rectangles were more beautiful than others. The closer the ratio of the dimensions was to a special number called the 'golden mean' or 'golden ratio', the more beautiful the rectangle. The Parthenon is an example of a building that was designed based on the golden ratio. Therefore, the rectangular outline of the Parthenon is a golden rectangle – a rectangle with dimensions that have a ratio equal to the golden mean."

1. Tell the students they will use K'NEX to create rectangles which are closer and closer to a golden rectangle. "The dimensions of these rectangles form a special sequence of numbers called the Fibonacci numbers. We will use these numbers to approximate the golden mean."

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Ask students to list examples of rectangles they encounter that have dimensions that are consecutive Fibonacci numbers.



Assessment:

Observe the students during the group work. Use a checklist to record whether students are using correct terminology and reasoning mathematically. Each group should receive a group grade on the activity. Ask the students to explain what they learned during the lesson in their Math Journal. Students should also record any concepts or ideas that are not clear to them at this time.

Extensions:

Ask students to do an online search of "Fibonacci numbers" to find examples of Fibonacci fractions in nature. Ask students to research additional information related to the golden mean. If groups combine their efforts to construct a very large rectangle, ask them to transfer it to a large sheet of paper. Students can then plot the golden spiral (use a different color) line on their drawing.



Fibonacci Numbers: Sequences and Ratios; Recursive Definition

In this activity you will generate a sequence of numbers called the **Fibonacci numbers**. You will create a recursive formula to find the next Fibonacci number in a series, find ratios of consecutive Fibonacci numbers, identify patterns, and create a recursive formula to find the next ratio of consecutive Fibonacci numbers.

1. You will use squares to create a sequence of rectangles that grow larger and larger. The following rules are used in generating the rectangles.

Start with one small square and at each successive step you add a square with a side length equal to the largest side of the previous rectangle. Following the rules in this sequence:

- A. Make a starter square, using 4 green rods and 4 white connectors.
- B. Add one square of the same size to the right edge of the starter square.
- B. Now add another square, with sides equal to the length of the two smaller squares, to the top edge of those two squares.
- C. Add the next larger size square to the left edge of the previous squares.
- D. And another larger square to the bottom edge of the previous squares.
- E. Repeat this counter clockwise pattern as you add each successive square.

The number associated with each square that makes up your rectangle will be the length of a side of the square, thus:

- Number 1 is associated with the 1 x 1 square
- Number 2 with the 2 x 2 square
- Number 3 with the 3 x 3 square, etc.

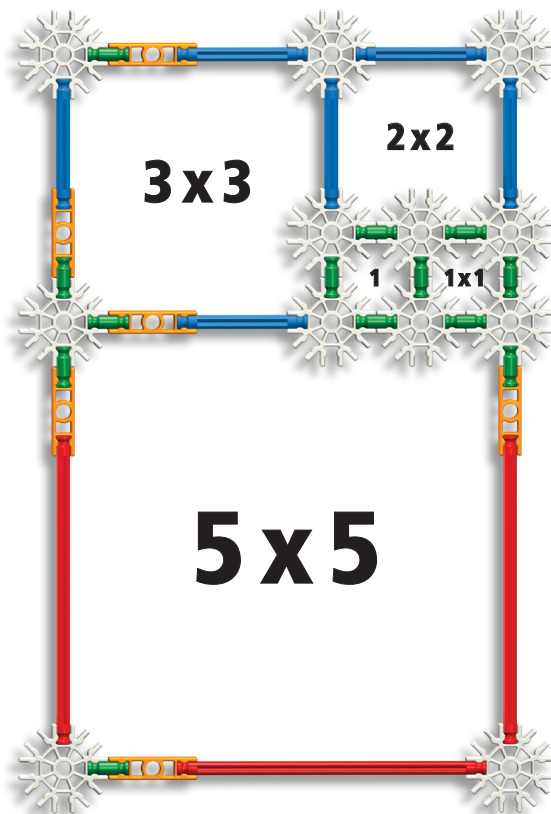
Use different combinations of these rods to make the correct size of the squares needed:

- Green rod segments (length = 1)**
- Blue rod segments (length = 2)**
- Red rod segments (length = 4)**

For example, the side of a square with length 5 will consist of a red segment plus a green segment with an orange connector between them and endpoints of white connectors.

Use this approach to create as large a rectangle as you can with the K'NEX available to you. The rectangle formed after adding 4 squares to the original square is shown below. (Also on page 22 of the Instructions Booklet.)

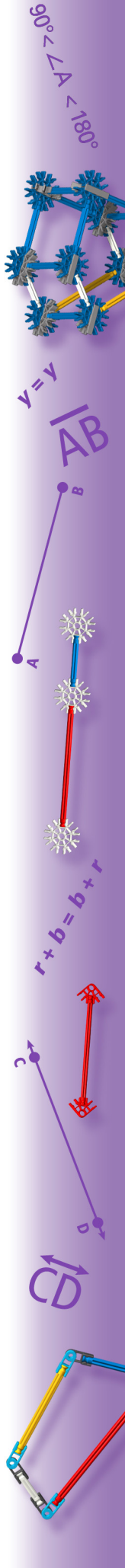
This would be a perfect opportunity to combine groups to see how large of a rectangle you can make collectively. Find a large area of floor space and you can have fun growing the rectangle as much as possible. Take a picture to show to other classes in the future; see who can build the biggest one.



The squares with dimensions noted were the ones added in the order: 1 x 1, 2 x 2, 3 x 3 and 5 x 5. Notice that the squares were added on the right, above, left, and below as required, so the next square will be added on the right.

Look at your final rectangle. The numbers associated with the squares in the order they were created after 4 squares were added is: 1, 1, 2, 3, 5 as shown above.

2. Add two more terms to this sequence using the squares you added to create the final rectangle. List the first seven terms of the Fibonacci numbers:



LESSON 7

3. Look at the Fibonacci numbers. Do you see a pattern to the sequence? Describe the pattern.

4. Use your pattern to predict the next Fibonacci number. How could you check if your pattern produces the correct answer?

Notation:

$F_1 = 1$ will represent the 1st Fibonacci number
 $F_2 = 1$ will represent the 2nd Fibonacci number
 $F_3 = 2$ will represent 3rd Fibonacci number, etc.
 Then F_n represents the n th Fibonacci number

5. Write an equation that shows how you could find F_3 using the values of F_1 and F_2 .

6. Write an equation that shows how you could find F_6 using the values of F_4 and F_5 .

7. Write an equation that shows how you could find F_n using the values of F_{n-2} and F_{n-1} .

The previous equation is a **recursive formula** that can be used to find any finite number of Fibonacci numbers desired.

When creating the (Fibonacci) rectangles you have rectangles with dimensions 1×1 , 1×2 , 3×2 , 5×3 , etc.

If we find the ratios of the sides of these rectangles we get the following sequence: $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}$, etc.

8. Add four more terms to this sequence of ratios (below).

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3},$$

9. Look at the sequence of ratios. Describe as many patterns as you can find in this sequence.

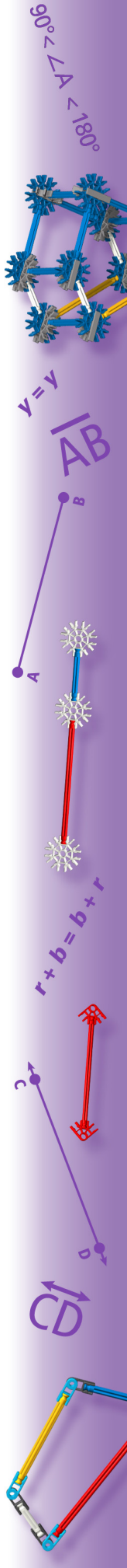
10. Use one or more of these patterns to predict what the ninth ratio will be.

The ratios you found are ratios of consecutive Fibonacci numbers. For example: $\frac{F_2}{F_1} = \frac{1}{1}$ and $\frac{F_3}{F_2} = \frac{3}{2}$.

11. Write an equation that shows how you could find the ratio $\frac{F_4}{F_3}$ using the numbers in the ratios $\frac{F_2}{F_1}$ and $\frac{F_3}{F_2}$.

12. Write an equation that shows how you could find $\frac{F_6}{F_5}$ using $\frac{F_4}{F_3}$ and $\frac{F_5}{F_4}$.

13. Write an equation that shows how you could find $\frac{F_n}{F_{n-1}}$ using $\frac{F_{n-2}}{F_{n-3}}$ and $\frac{F_{n-1}}{F_{n-2}}$.

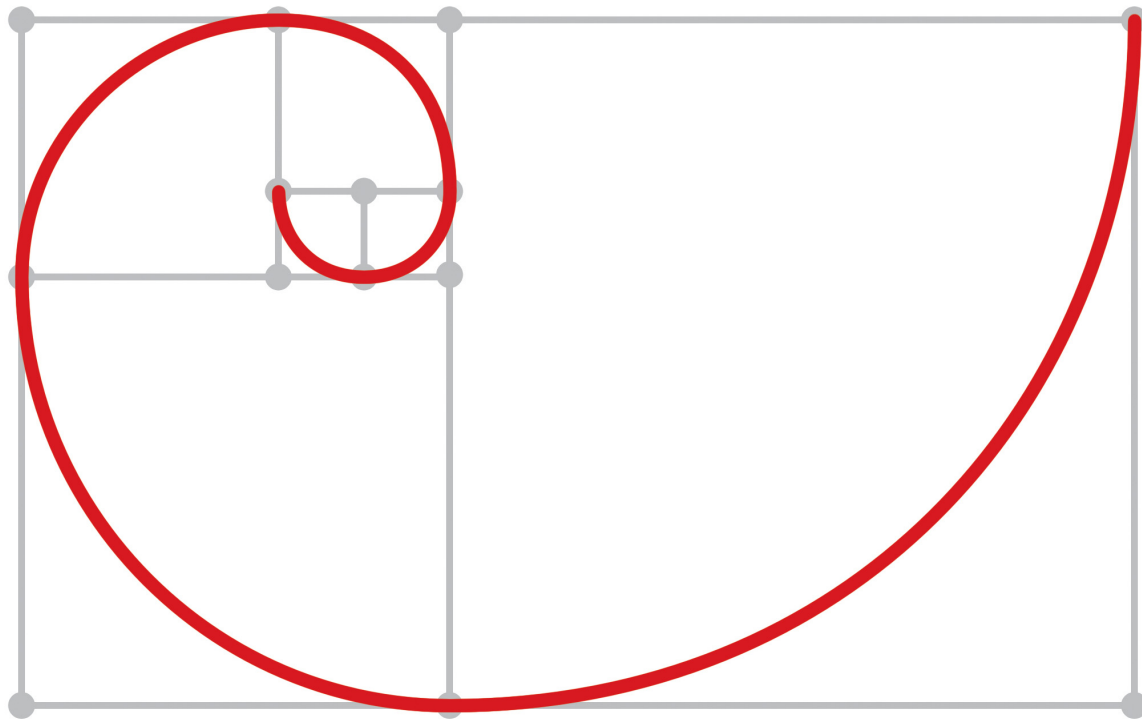


14. Use a calculator to find decimal approximations for the first nine Fibonacci ratios rounded to the nearest thousandth. The first few are done for you.

$$\frac{1}{1} = 1 \quad \frac{2}{1} = 2 \quad \frac{3}{2} = 1.5$$

15. Look at the decimal approximations of the Fibonacci ratios. What do you notice?

16. The number these ratios are approaching is called the golden ratio. Use the Internet to learn more about the golden ratio.

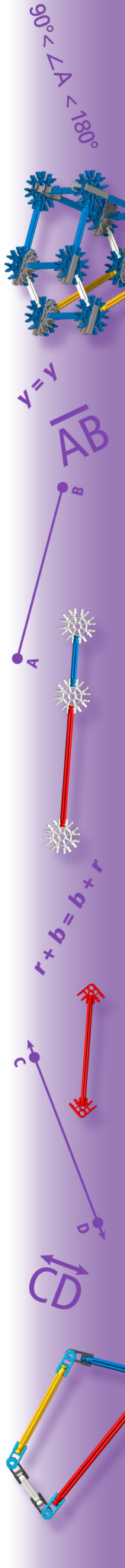


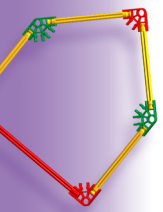
Fibonacci Spiral:

If you connect the upper left vertex of the first square with the lower right vertex with an arc, then connect this vertex of the second square with the opposite vertex with an arc, and continue in this manner through all of the squares, you will create a spiral. See the figure above.

As each of the vertices in your model has a circular hole to represent endpoints, you can transfer your model to a large sheet of easel or drawing paper with the use of a pencil and straight edge or ruler.

17. Complete the spiral with pencil, marker, or crayon once you have each of the sections of the model on paper.





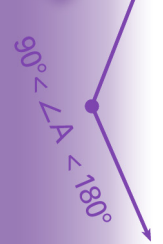
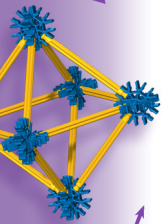
EF



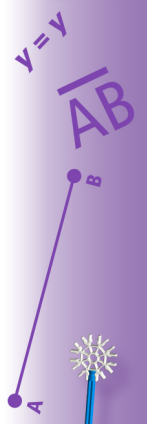
$$y + s = s + y$$

$$y + y + w + w = 2y + 2w$$

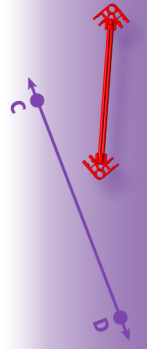
$$w < r$$



$$90^\circ < \angle A < 180^\circ$$



$$r + b = b + r$$



CD



LESSON 8

Addition & Subtraction of Fractions

Lesson Topics: Adding and subtracting fractions, developing rules.

Lesson Length: Two 50-minute periods

Student Objectives:

Students will:

- Understand what it means for two fractions to be equivalent.
- Develop a rule for adding two fractions.
- Develop a rule for subtracting two fractions.
- Use inductive reasoning to make predictions.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX to create a line segment of unit length and various line segments with fraction lengths.
- They will then explore which fractions are equivalent because the associated line segments have the same length (are congruent). These line segments will then be added by linking them to form larger line segments.
- Students will find the sum of the two fractions by determining the length of the new line segment. Then they will look for a pattern to the answers and use the pattern to create a rule for adding two fractions.
- Subtraction of fractions will be modeled as taking one line segment away from another line segment. The length of the remaining segment represents the difference of the two fractions represented by the original line segment.
- Students will find a pattern to the answers obtained and use the pattern to create a rule for finding the difference of two fractions.

Materials and Equipment:

- **K'NEX rods:**
 - 4 red
 - 10 blue
 - 16 green
- **K'NEX connectors per group:**
 - 12 white
 - 11 yellow
 - 10 orange (straight)
- **Copies of the Lesson #8 Student Inquiry Sheets**

A – Motivation and Introduction:

1. "Jimmy says he can eat $\frac{1}{2}$ of a pizza. Dean says he can eat $\frac{7}{12}$ of a pizza. If two pizzas were ordered for the basketball group, how much is left for the other members of the group? In this activity you will learn how to solve problems such as this one."
2. Inform the class that they will use K'NEX to create line segments of various lengths. Students will be shown how the length of the segment is the distance from the center of one end connector to the center of the other end connector. The class will create a line segment composed of 12 green segments, two white connectors (to represent the endpoints), and 11 orange and yellow connectors. They will be told that the length of this line segment represents the number 1. The value of other line segments will depend on how many have to be combined to make a line segment of length 1.

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Review the rules developed for adding and subtracting fractions. Present one addition and one subtraction problem to the entire class with 4 possible answers offered for each question. Use polling with A, B, C, D cards to determine if students can effectively use the rules of adding and subtracting fractions to find the sum and difference of two fractions.

Assessment:

Observe the students during the group work. Use a checklist to record whether students are using correct terminology and reasoning mathematically. Each group should receive a group grade on the activity. Ask the students to explain what they learned during the lesson in their Math Journal. Students should also record any concepts or ideas that are not clear to them at this time.

Extensions:

Students are asked to use what was learned in this activity to solve problems involving whole numbers and mixed numbers.



Addition and Subtraction of Fractions

Note: For this activity a line segment will be represented by rods and connectors. Only the two endpoints of the segment will be represented by white connectors. We will use orange and yellow connectors for points inside the segment.

Create a line segment using twelve green rods, two white connectors (for the endpoints of the segment) and eleven orange or yellow connectors for the middle connections. This will be our unit segment. The length of this segment is 1.

Create a line segment using one green rod and two white connectors. Since it took twelve of these segments to make the segment of length 1, the length of this segment is one-twelfth.

Therefore one green rod with connectors = 1 twelfth

Create a segment using one blue rod and two white connectors.

1. How many blue rods with connectors would be needed to equal the length of the unit segment?

2. What is the length of this segment as a fraction of the unit length?

1 blue rod with connectors =

Create a segment using one blue rod, one green rod, two white connectors and one orange connector.

3. How many of these line segments would be needed to equal the length of the unit segment?

4. What is the length of this segment as a fraction of the unit length?

1 blue plus 1 green rod with connectors =

Create a segment using one red rod and two white connectors.

5. How many of these line segments would be needed to equal the length of the unit segment?

6. What is the length of this segment as a fraction of the unit length?

1 red rod with connectors =

Create a segment using one red rod, one blue rod, two white connectors and one orange connector.

7. How many of these line segments would be needed to equal the length of the unit segment?

8. What is the length of this segment as a fraction of the unit length?

1 red plus 1 blue rod with connectors =

Equivalent Fractions:

Using our unit length we say two fractions are **equivalent** if they have the same length. For example, 6 twelfths equals 1 half, since they have the same length. Using the fraction lengths from page 1, what other fraction(s) are equivalent to 1 half?

6 twelfths = 1 half _____ = 1 half

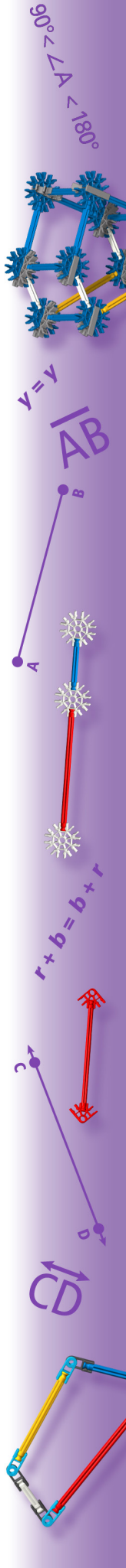
_____ = 1 half _____ = 1 half

9. Look at the fractions that are equal to 1 half. Do you see a pattern? Describe the pattern in your own words.

10. Use the pattern to predict another fraction that will equal 1 half.

11. What fraction is equivalent to 1 third?

12. Predict another fraction that will equal 1 third.



13. What must be true about the number of line segments needed to make one third versus the number of segments needed to make the unit length?

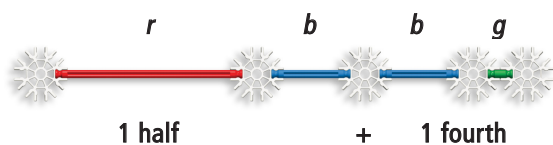
Connect two 1 third fraction models to create a line segment with a length of 2 thirds.

14. What fraction is equivalent to 2 thirds?

15. Predict two other fractions that will equal 2 thirds.

Addition of Fractions:

We can indicate the sum of two fractions by connecting the fractions together. For example, the sum 1 half plus 1 fourth would be represented by the segment found by combining a red and blue segment (1 half) and a blue and green segment (1 fourth). See the figure below.



16. What fractions are equivalent to this sum?

17. Which of the two equivalent fractions requires the fewest segments? We say this is the sum of 1 half plus 1 fourth in simplest form.

In order to make it easier to work with fractions and to see patterns, we represent fractions using fraction notation. For example:

1 half is represented symbolically as $\frac{1}{2}$

And 2 thirds is represented symbolically as $\frac{2}{3}$

The top number, called the **numerator**, is the number of segments of the appropriate length used to make the fraction. The bottom number, called the **denominator**, is the name of the fraction. It indicates how many of these segments are needed to make the unit length. Thus, the addition just performed can be written symbolically as:

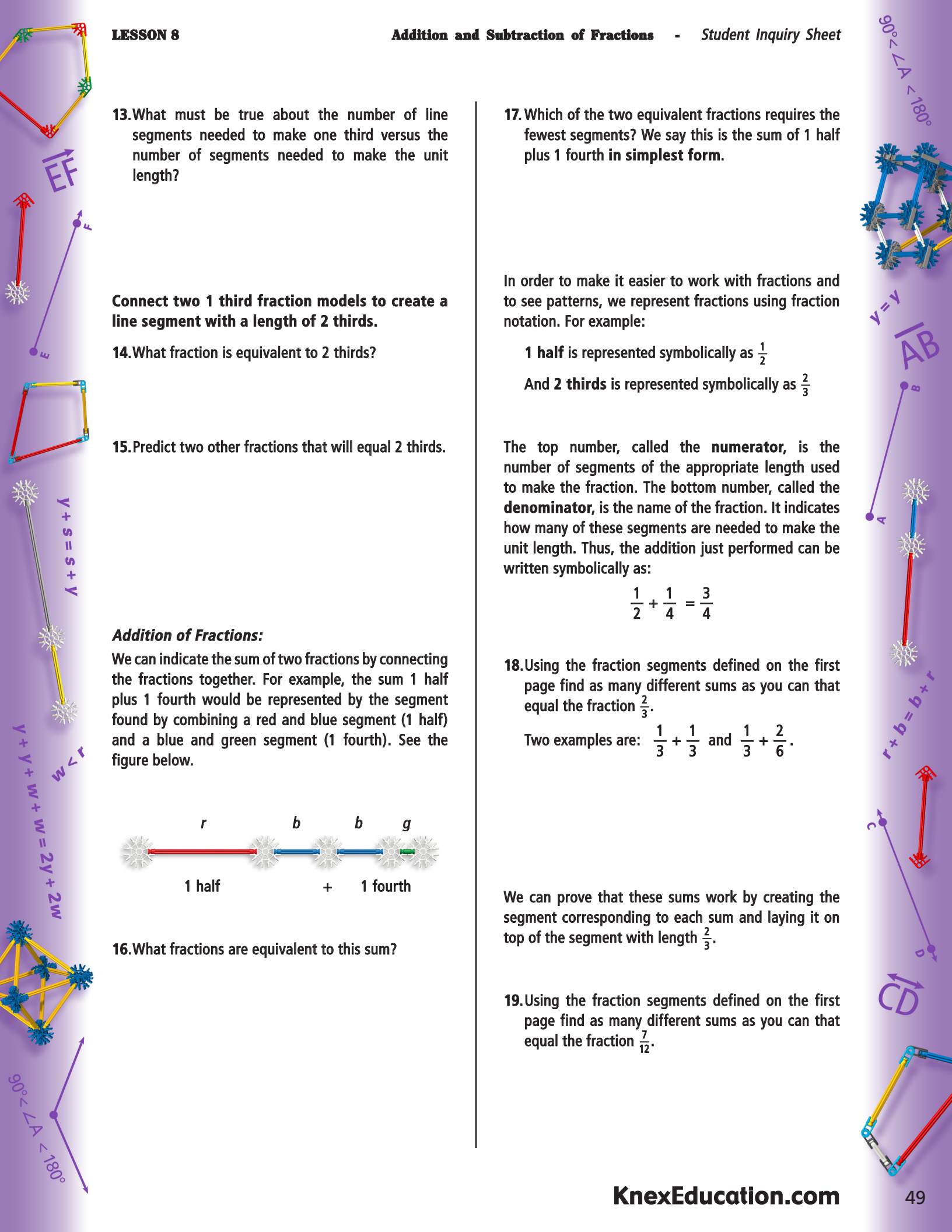
$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

18. Using the fraction segments defined on the first page find as many different sums as you can that equal the fraction $\frac{2}{3}$.

Two examples are: $\frac{1}{3} + \frac{1}{3}$ and $\frac{1}{3} + \frac{2}{6}$.

We can prove that these sums work by creating the segment corresponding to each sum and laying it on top of the segment with length $\frac{2}{3}$.

19. Using the fraction segments defined on the first page find as many different sums as you can that equal the fraction $\frac{7}{12}$.



19. Using the fraction segments defined on the first page find as many different sums as you can that equal the fraction $\frac{3}{2}$.

Note: The fraction $\frac{3}{2} = 1\frac{1}{2}$, so the length of the fraction is more than 1. Any fraction with length greater than or equal to 1 is called an **improper fraction**. There is nothing wrong with writing fractions of this type. The name just means we could write the fraction as either a whole number or a **mixed number** – a sum of a whole number and a fraction

20. Using the fraction segments defined on the first page, find the following sums. Write each answer in simplest form. Do not change improper fractions to mixed numbers.

A. $\frac{1}{2} + \frac{1}{3} =$

D. $\frac{2}{3} + \frac{3}{4} =$

B. $\frac{1}{4} + \frac{1}{3} =$

E. $\frac{3}{2} + \frac{2}{3} =$

C. $\frac{2}{3} + \frac{1}{2} =$

F. $\frac{3}{4} + \frac{4}{3} =$

Look at your answers to the sums above.

21. How could you determine the value of the denominator of the sum from the denominators of the fractions being added?

22. How could you get the numerator of the sum by using multiplication and addition with the four numbers (numerator and denominator of each fraction) of the problem?

23. Use this pattern to find the sum of:

$$\frac{1}{4} + \frac{2}{3} =$$

Note: Check your answer using the line segments defined on the first page.

24. Use this pattern to find the sum of:

$$\frac{3}{4} + \frac{1}{6} = \quad =$$

Note: The answer will not be in simplest form. Simplify your answer. Check your answer using the line segments defined on the first page.

25. Does the pattern hold true when the denominators have a common factor?

26. Use this pattern to find the sum:

$$\frac{5}{12} + \frac{1}{3} = \quad =$$

We can represent this method symbolically as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

- We can see why the method works by using equivalent fractions.
- Consider the sum $\frac{1}{4} + \frac{2}{3}$
- The fraction $\frac{1}{4} = \frac{1}{4} \cdot \frac{3}{3} = \frac{3}{12}$, while $\frac{2}{3} = \frac{2}{3} \cdot \frac{4}{4} = \frac{8}{12}$
- Since 3 twelfths plus 8 twelfths equals 11 twelfths, this is our sum. Notice that the numerators could be found by cross multiplication, as indicated by our pattern.



Subtraction of Fractions:

We can use the fraction line segments made from K'NEX that were defined on the first page to subtract fractions as well as add fractions. We simply align the right endpoints of the two segments representing the two fractions. The answer to the subtraction problem is how much is left when the second fraction length is taken away from the first fraction length.

The description below outlines the subtraction $\frac{1}{2} - \frac{1}{3}$.

- The fraction $\frac{1}{2}$ is represented by the red and blue rod segment.
- The fraction $\frac{1}{3}$ is represented by the red rod segment.
- The amount left has length equal to a blue rod segment, which is the fraction $\frac{1}{6}$.

27. Use K'NEX pieces and the fraction segments defined on the first page to model the following differences.

A. $\frac{3}{4} - \frac{2}{3} =$

D. $\frac{3}{4} - \frac{1}{3} =$

B. $\frac{2}{3} - \frac{1}{2} =$

E. $\frac{2}{3} - \frac{1}{4} =$

C. $\frac{1}{3} - \frac{1}{4} =$

F. $\frac{3}{2} - \frac{1}{3} =$

28. Does a similar pattern hold true for subtracting fractions?

29. If so, describe the pattern.

30. Use the pattern to find the sum:

$$\frac{5}{4} - \frac{2}{3} =$$

Note: Use the fraction line segments to check if the answer is correct.

31. Does this pattern work to find the sum:

$$\frac{3}{4} - \frac{5}{12} =$$

32. Explain your reasoning.



LESSON 9

Multiplication & Division of Fractions

Lesson Topics: Multiplying fractions, dividing fractions, and developing rules.

Lesson Length: Two 50-minute periods

Student Objectives:

Students will:

- Understand that multiplication can be interpreted as the area of a rectangle with dimensions equal to the two factors.
- Develop a rule for multiplying two fractions.
- Understand that division can be interpreted as the missing factor in a multiplication problem.
- Develop a rule for dividing two fractions.
- Use inductive reasoning to make predictions.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX to create a square with area equal to 1 square unit, the unit square. They will then create rectangles where the dimensions correspond to the fractions being multiplied. For example, $\frac{1}{2} \cdot \frac{1}{3}$ will be represented by a rectangle with a length of $\frac{1}{2}$ (red and blue rods with 3 connectors) and width of $\frac{1}{3}$ (red rod with two connectors). The area of this rectangle will be the desired answer. Since 6 of these rectangles will fit in the unit square, its area (the desired product) is $\frac{1}{6}$.
- After finding several products of fractions, the students will look for a pattern to the answers and use the pattern to create a rule for multiplying two fractions. Division of fractions will be modeled using the missing factor model for division. For example, $\frac{1}{6} \div \frac{1}{3} = ?$ is equivalent to $\frac{1}{3} \cdot ? = \frac{1}{6}$. Thus, we know the area of a rectangle and we know the length is $\frac{1}{3}$. The quotient is the width of the rectangle.
- After doing several division problems using this approach, students will find a pattern to the answers obtained and use the pattern to create a rule for finding the quotient of two fractions.

Materials and Equipment:

- **K'NEX rods:**
20 red
10 blue
10 green
- **K'NEX connectors per group:**
4 blue
14 white
11 yellow
10 orange
- Pens and/or pencils
- Copies of the Lesson #9 Student Inquiry Sheets

A – Motivation and Introduction:

1. "Three students buy half of a pizza. They each want one-third of half the pizza. What fraction of a whole pizza will each student receive?"

"A pizza was divided into twelfths. Half of the pizza was left after lunch. How many slices of pizza are left?"

These are examples of problems the students will learn to solve in this activity.

2. The class reviews line segments with length $\frac{1}{12}$, $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$. The area model for finding a product is

reviewed. The class is told that they will use K'NEX to create a unit square that is 3 red rods plus connectors on each side. The teacher then illustrates how students can determine the area of a rectangle in terms of this unit area. The concept of modeling division as a missing factor in a multiplication problem will be explained using whole numbers.

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.

2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Review the rules developed for multiplying and dividing fractions. Present one multiplication and one division problem to the entire class with 4 possible answers for each question. Use polling with A, B, C, D cards to determine if students can use the rules for the multiplication and division of fractions to find the product and quotient of two fractions.

Assessment:

Observe the students during the group work. Use a checklist to record whether students are using correct terminology and reasoning mathematically. Each group should receive a group grade on the activity. Ask the students to explain what they learned during the lesson in their Math Journal. Students should also record any concepts or ideas that are not clear to them at this time.

Extensions:

Students are asked to use what was learned in this activity to solve problems involving whole numbers and mixed numbers.



Multiplication and Division of Fractions

Multiplication:

We will use the area model to represent the product (answer to a multiplication problem) of fractions. We will use the same segment lengths as in the Addition and Subtraction of Fractions activity for the values of different fractions. To summarize what we used in that activity:

- Green rod and two connectors: $\frac{1}{12}$
- Blue rod and two connectors: $\frac{1}{6}$
- Blue and green rod with connectors: $\frac{1}{4}$
- Red rod and two connectors: $\frac{1}{3}$
- Red and blue rod with connectors: $\frac{1}{2}$

Thus, 3 red rods connected by 4 white connectors will have length 1.

1. Explain why?

A square made with sides of length 1 (for example, 3 red rods plus connectors) will have an area of 1 square unit. Create a square using 12 red rods and 12 connectors. Place the square on a large sheet of paper and, use a pen or pencil to mark the four vertices, and use a ruler to transfer the square to the paper. This will be used to determine the value of a product of two fractions.

We can represent the multiplication problem $\frac{1}{2} \cdot \frac{1}{2}$ as the area of the rectangle with sides of length $\frac{1}{2}$. Create a rectangle where each side has length $\frac{1}{2}$ (red and blue rods with connectors).

2. Lay the rectangle just created in one corner of the unit square marked on the paper. Mark the vertices of the rectangle in pencil. Determine how many of these rectangles it would take to fill up the unit square. The number needed tells us the value of the area of the rectangle. For example, if it takes 3 of these rectangles to fill the unit square, the value is $\frac{1}{3}$, while if it takes 6 rectangles to fill the unit square, the value is $\frac{1}{6}$.

$$\frac{1}{2} \cdot \frac{1}{2} =$$

3. Use the technique described above to find the product of these multiplied fractions.

- A. $\frac{1}{2} \cdot \frac{1}{4} =$
- B. $\frac{1}{3} \cdot \frac{3}{4} =$
- C. $\frac{1}{4} \cdot \frac{2}{3} =$
- D. $\frac{2}{3} \cdot \frac{3}{4} =$

Now consider the previous multiplication problem D. $\frac{2}{3} \cdot \frac{3}{4}$. If we multiply the two numerators and the two denominators and put the product of the numerators over the product of the denominators we have $\frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12}$.

- 4. Compare this answer with the one you found using the K'NEX. Are the two answers equivalent fractions?
- 5. Try this with the other multiplication problems above. Do you always get an equivalent fraction?

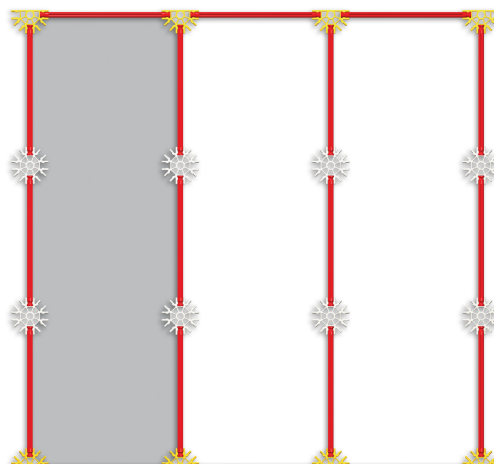
This leads to the following rule for multiplying fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$.

6. Use this rule to find the product of these multiplied fractions. Write the answers in simplest form.

- A. $\frac{5}{12} \cdot \frac{2}{5} =$
- B. $\frac{4}{3} \cdot \frac{7}{12} =$
- C. $\frac{5}{6} \cdot \frac{3}{10} =$
- D. $\frac{2}{3} \cdot \frac{3}{8} =$

Division:

Since the answer to the multiplication problem $\frac{1}{2} \cdot \frac{3}{4}$ can be interpreted as the area of the rectangle with sides of length $\frac{1}{2}$ and $\frac{3}{4}$, the division problem $\frac{1}{6} \div \frac{1}{3}$ can be interpreted as meaning the area of a rectangle with one side of length $\frac{1}{3}$ unit is $\frac{1}{6}$ square units. The answer to the division problem, called the **quotient**, can be interpreted as the length of the adjacent side of the rectangle. Because the one side has length $\frac{1}{3}$, we will divide the unit square into 3 equal regions as shown below. The shaded region shows a $\frac{1}{3}$ by 1 rectangle.



In order to get a rectangle with area $\frac{1}{6}$ we need to divide this unit square into 6 congruent rectangles. Then each rectangle will have area $\frac{1}{6}$. To do this you will have to replace each of the center, vertical red rods with two blue rods joined by a blue connector.

7. At what point(s) could you draw one or more horizontal lines to obtain 6 congruent rectangles?

The other dimension of one of these rectangles is the desired quotient.

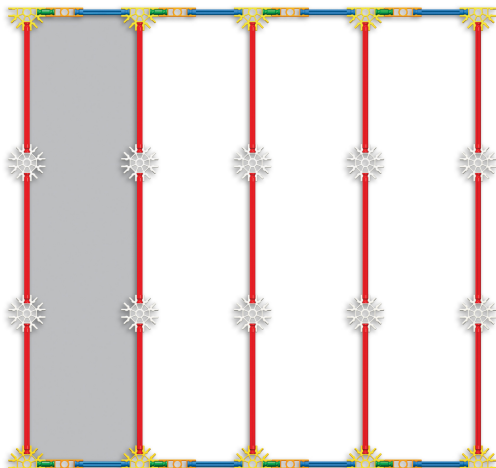
8. What is the quotient? You can use the K'NEX fraction lengths to determine the quotient.

$$\frac{1}{6} \div \frac{1}{3} =$$

9. Check the answer just found by multiplying the quotient and $\frac{1}{3}$. Do you get $\frac{1}{6}$?

10. What is the interpretation of the division problem $\frac{1}{12} \div \frac{1}{4}$ in terms of the K'NEX?

The figure below shows the desired width via the shading. You will have to change to blue and green rods plus connectors to represent $\frac{1}{4}$ on the top and bottom sides of the unit square.



11. At what point(s) could you draw one or more horizontal lines to obtain 12 congruent rectangles? What is the second dimension of one of these rectangles? This is the desired quotient.

$$\frac{1}{12} \div \frac{1}{4} =$$

12. Use this procedure to find $\frac{1}{18} \div \frac{1}{6}$. Show how you solved the problem.

Look at the results of the division problems so far.

$$\frac{1}{6} \div \frac{1}{3} = \frac{1}{2} = \frac{3}{6}$$

$$\frac{1}{12} \div \frac{1}{4} = \frac{1}{3} = \frac{4}{12}$$

$$\frac{1}{18} \div \frac{1}{6} = \frac{1}{3} = \frac{6}{18}$$

13. How could you have obtained the quotient of $\frac{3}{6}$ from the original fractions?

14. How could you have obtained the quotient of $\frac{4}{12}$ from the original fractions?

15. How could you have obtained the quotient of $\frac{6}{18}$ from the original fractions?

This leads to the following rule for dividing fractions:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$$

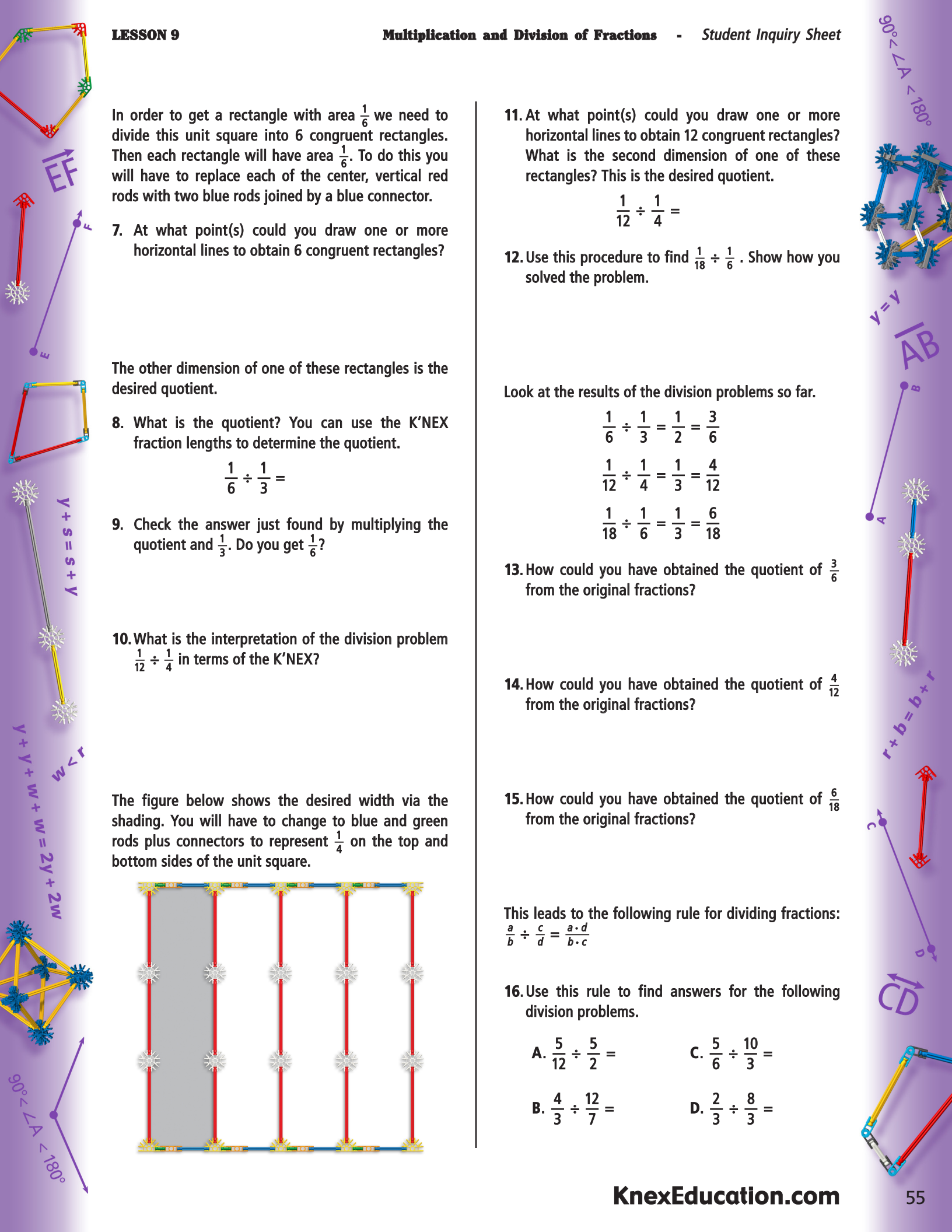
16. Use this rule to find answers for the following division problems.

A. $\frac{5}{12} \div \frac{5}{2} =$

C. $\frac{5}{6} \div \frac{10}{3} =$

B. $\frac{4}{3} \div \frac{12}{7} =$

D. $\frac{2}{3} \div \frac{8}{3} =$



LESSON 10

Properties of Numbers

Lesson Topics: Justifying Axioms for Rational Numbers

Lesson Length: One 50-minute periods

Student Objectives:

Students will:

- Understand the axioms of rational numbers.
- Recognize that the axioms of rational numbers are sensible.
- Feel comfortable explaining how the axioms can be used to simplify computations.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX to show that the reflexive, symmetric, and transitive properties of equality make sense.
- They will also use geometric models to help them understand what an identity is and what makes 0 the additive identity and 1 the multiplicative identity.
- They will understand that a set of numbers is dense, if given any two numbers in the set you can find another number in the set that is larger than the one number and smaller than the other. This will be accomplished by building rectangles with areas a , b and $\frac{a+b}{2}$ and showing that the area of the last rectangle is between the areas of the other two.
- Line segments will be used to show that the commutative and associative properties of addition are sensible. The areas of rectangles will be used to show that the commutative property of multiplication makes sense, while the volumes of two rectangular prisms will be used to show that the associative property of multiplication makes sense.
- The distributive property will be illustrated using rectangles.

Materials and Equipment:

- K'NEX Middle School Math set and Instructions Booklets
- Copies of the Lesson #10 Student Inquiry Sheets

A – Motivation and Introduction:

"If twenty nickels have the same value as four quarters and four quarters have the same value as one dollar, what can you say about the values of twenty nickels and one dollar? This is an example of the use of the transitive property of equality. In this activity we will use geometric models to show what several properties (or axioms) of rational numbers mean and why it makes sense to assume they are true without proof."

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.

3. Circulate among the groups, guiding them as they complete the project.

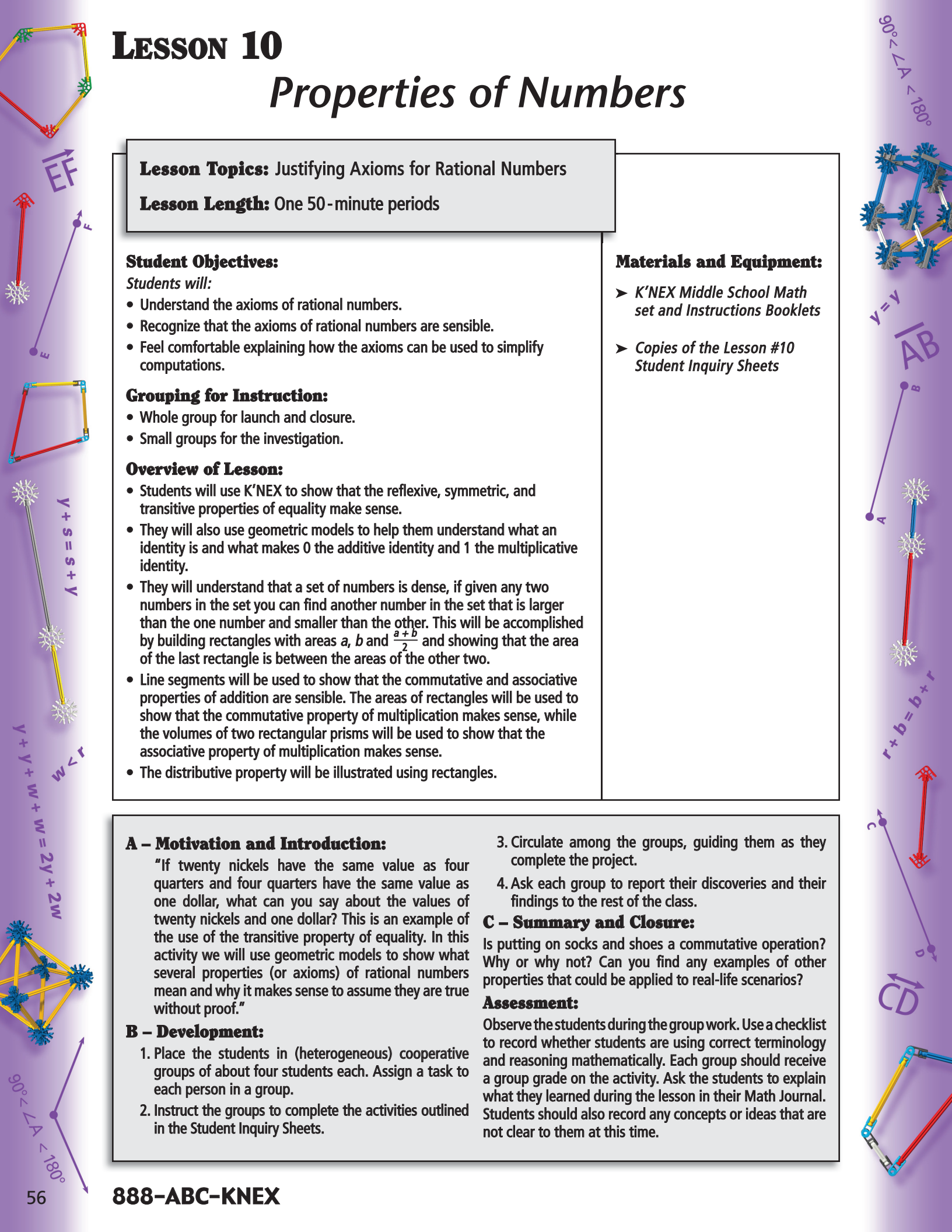
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Is putting on socks and shoes a commutative operation? Why or why not? Can you find any examples of other properties that could be applied to real-life scenarios?

Assessment:

Observe the students during the group work. Use a checklist to record whether students are using correct terminology and reasoning mathematically. Each group should receive a group grade on the activity. Ask the students to explain what they learned during the lesson in their Math Journal. Students should also record any concepts or ideas that are not clear to them at this time.



Extension:

Show the solution of an equation that involves several properties and ask students to identify the properties used.

A Volume of a Pyramid Challenge has been developed to explore students' understanding of the concepts from earlier lessons and to test their mathematics reasoning and problem solving skills. This exercise is located on the final page of the Student Inquiry Pages if you choose to use it with your students.

Teacher Notes:

Challenge Problem - Volume of a Pyramid

1. Show the students a cube that was made with silver rods (prepare the model before class). Be sure that the double blue or blue/gray connectors are positioned so you can connect rods at each vertex toward the centroid of the cube – the point in the middle of the cube. Ask the students to find the volume of this cube. ($V = s \cdot s \cdot s$ or $V = s^3$).
2. Ask students what the volume would be for a prism with the same base, but height that is half the height of the cube. ($V = s \cdot s \cdot y$ or $V = s^2y$ (as a y line segment is $\frac{1}{2}$ of an s line segment)).

3. Create 8 segments consisting of a yellow rod, blue rod, and a straight orange connector. Attach one segment at each vertex of the cube so it points toward the centroid of the cube.

- Ask students to imagine that they meet at the centroid. Show the students that a face of the cube and 4 of these segments form a square pyramid with height half of the length of a silver segment. Ask how many of these square pyramids were formed (6).
 - What is the volume of one of these pyramids? ($V = \frac{1}{6} s^3$.)
 - How do you know? (If the total volume is s^3 and there are 6 pyramids inside the cube then one has a volume of $\frac{1}{6} s^3$.)
4. Compare the volume of one of these pyramids with the volume of the "half cube". Note that the volume is one-third the volume of the prism with the same base and height as the pyramid. This is always true. In symbols, if B is the area of the base of a pyramid and h is its height, then the volume V is given by:

$$V = \frac{1}{3} Bh$$



Properties of Numbers

Conventions for this activity:

- A number can be represented by the length of a line segment, the area of a rectangle, or the volume of a rectangular surface (prism).
- Addition of two numbers can be shown by linking two line segments, two rectangles, or two rectangular surfaces to form a larger line segment, rectangle, or rectangular surface (prism).
- Multiplication of two numbers can be represented as the area of a rectangle with dimensions equal to the two numbers or as the volume of a prism with the area of the base equal to one number and the height equal to the other number.
- The number zero can be represented by a connector, since a point has 0 length, width, and height.
- The number 1 can be represented by a unit line segment, a 1 x 1 square, or a 1 x 1 x 1 cube.

In this activity we will use K'NEX shapes to help us "see" that the various properties of rational numbers (fractions, whole numbers, and mixed numbers) make sense.

Reflexive Property:

A number or number expression is equal to itself. In symbols, $a = a$.

1. Create two segments each consisting of 1 white rod and 2 white connectors. Do they have the same length?

2. Create two segments each consisting of 1 red rod and 2 white connectors. Do they have the same length?

3. Would this be true for any line segment?

Symmetric Property: If one number or number expression is equal to a second number or number expression, then the second number or number expression is equal to the first number or number expression. In symbols, if $a = b$, then $b = a$.

4. Show that the line segment consisting of 2 green rods and 3 connectors has the same length as the line segment consisting of 1 blue rod and 2 connectors by putting the second segment on top of the first. This demonstrates that the number represented by the first segment equals the number represented by the second segment. If you place the first segment on top of the second segment, will they still be equal in length? If so, then the number represented by the second segment equals the number represented by the first segment.

5. Would this equality hold if we used other equal segments such as 2 white rods with 3 connectors compared with 1 yellow rod with 2 connectors? (Refer to page 13 of the Instructions Booklet for additional equalities. Can you find other combinations of rods and connectors that would be equal to the segment made up of a blue and a red rod or the segment made up of a yellow and a silver rod that are found on page 13? How many can you find? List them below. (i.e., $r + b = 3b = 6g = \text{etc.}$)

Transitive Property:

If one number or number expression is equal to a second number or number expression and the second number or expression was equal to a third number or expression, then the first number or expression will equal the third number or expression. In symbols, if $a = b$ and $b = c$, then $a = c$.

6. Show that the line segment (a) consisting of 4 white rods and 5 connectors has the same length as the line segment (b) consisting of 2 yellow rods and 3 connectors. That is, the associated numbers are equal.



7. Show that the line segment (b) consisting of 2 yellow rods and 3 connectors has the same length as the line segment (c) consisting of 1 silver rod and 2 connectors. That is, the associated numbers are equal.

8. Show that the original line segment (a) consisting of 4 white rods and 5 connectors has the same length as the final line segment (c) consisting of 1 silver rod and 2 connectors. That is, the associated numbers are equal.

Therefore, using your models you should be able to demonstrate that $a = b$, then $b = c$ and also that $a = c$. You have used your models to demonstrate the transitive property.

9. Would this work with other segments? Model another example of the Transitive Property using K'NEX pieces. Describe the example you modeled.

Additive Identity:

The number 0 has the property that any number plus 0 equals 0 plus that number which equals that number. In symbols, $a + 0 = 0 + a = a$.

If you combine a connector (representing a point) with a line segment you will get the same line segment whether the connector replaces the left connector or right connector of the line segment.

Multiplicative Identity:

The number 1 has the property that any number times 1 equals 1 times that number which equals that number. In symbols, $1 \cdot a = a \cdot 1 = a$.

Let the line segment composed of a blue rod and two white connectors be defined as the unit length (1). Then $1 \cdot a$ is the area of a rectangle that with length 1 unit and width a units. Let the length of the line segment composed of a yellow rod and two white connectors have length a .

10. Create a rectangle with two sides made of blue rods with white connectors and two sides with the yellow rod with two white connectors. The area of this rectangle is $1 \cdot a$. The expression $1 \cdot a$ can be represented by the area of a rectangle with length a and width 1.

11. Create a rectangle with two sides the yellow rod with two white connectors and the other two sides blue rods with two white connectors. The area of this rectangle is $a \cdot 1$. Show that the two rectangles are congruent, so their areas are equal. Also, note that a unit squares will fit inside the rectangle, so the area is a .

Density:

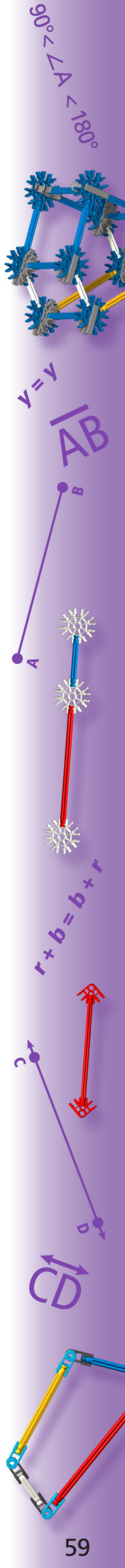
The rational numbers are dense, meaning that between any two rational numbers a and b you can find another rational number, $\frac{a+b}{2}$. In symbols, if a and b are rational numbers with $a < b$, then $\frac{a+b}{2}$ is a rational number with $a < \frac{a+b}{2} < b$.

We first note that the definition of addition of rational numbers guarantees that $a + b$ is a rational number. Since dividing by 2 is the same as multiplying by $\frac{1}{2}$, $\frac{a+b}{2}$ has to be a rational number.

We use rectangles to show the inequality. Again, let the line segment composed of a blue rod and two connectors have length 1. Then the line segment composed of one green rod and two connectors has length $\frac{1}{2}$, because it takes two of these lengths to make the line segment of length 1. Let the line segment composed of one yellow rod and two connectors have length a . Finally, let the line segment composed of one red rod and two connectors have length b .

12. Create a rectangle with length 1 and width a . What is the area of this rectangle? (See Multiplicative Identity.)

13. Create a rectangle with length 1 and width b . What is the area of this rectangle?



14. Compare the two rectangles. Is $a < b$? Justify your answer.

15. Create a rectangle with length $\frac{1}{2}$ and width $a + b$. What is the area of this rectangle?

16. Now create a rectangle with length $\frac{1}{2}$ and width $2b$. What is the area of this rectangle?

17. Create a rectangle with length $\frac{1}{2}$ and width $2b$. What is the area of this rectangle?

18. Compare the three rectangles with length equal to $\frac{1}{2}$. Is $a < \frac{a+b}{2} < b$? Explain how you know.

Commutative Property of Addition:

If two rational numbers are added, the order in which they are added does not affect the sum. In symbols, $a + b = b + a$.

19. Create the sum $a + b$ by linking the line segment composed of a yellow rod and two connectors with the line segment composed of a silver rod and two connectors.

20. Create the sum $b + a$ by reversing the order in which you combine the two line segments. Compare the two segments. Does $a + b = b + a$?

(Refer to page 13 in the Instructions Booklet for another example of the commutative property of addition using K'NEX.)

Commutative Property of Multiplication:

If two rational numbers are multiplied, the order in which they are multiplied does not affect the product. In symbols, $a \cdot b = b \cdot a$.

Let the line segment composed of one yellow rod and two connectors have length a and let the line segment composed of a red rod and two connectors have length b .

21. Create a rectangle with length a and width b . What is the area of this rectangle?

22. Create a rectangle with length b and width a . What is the area of this rectangle?

23. Compare the two rectangles. Are they congruent? What does this mean about their areas?

Associative Property of Addition:

When adding three numbers the result will be the same if you find the sum of the first two and then add the third or find the sum of the second two and then add the first. In symbols, $(a + b) + c = a + (b + c)$.

Let the line segment composed of one yellow rod and two connectors have length a and let the line segment composed of one red rod and two connectors have length b . Let the length of the line segment composed of one silver rod and two connectors be c . Connect one line segment of length a with one of length b to get a line segment with length $a + b$. Then combine this line segment with one line segment of length c . The length of this line segment is $(a + b) + c$.

24. Connect one line segment of length b with one line segment of length c . What is the length of this new line segment?

25. Connect this new line segment with one line segment of length a . What is the length of this combined line segment?



26. Compare the two line segments you created. Does $(a + b) + c = a + (b + c)$? Explain your reasoning.

Associative Property of Multiplication:

When multiplying three numbers the result will be the same if you find the product of the first two and then multiply by the third or find the product of the second two and then multiply by the first. In symbols, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

27. Create a rectangle with length a (yellow rods) and width b (red rods) (the vertices of the rectangle should be made of blue and gray connectors slid together as shown in the Instructions Booklet). What is the area of this rectangle?

28. Use this rectangle as the base of a prism with height c (silver rods). Create the prism. What is the volume of this prism? (Hint: The volume of a prism equals the area of the base times the height.)

29. Create a rectangle with length b and width c (the vertices of the rectangle should be made of blue and gray connectors slid together as shown in the Instructions Booklet).

30. What is the area of this rectangle?

31. Use this rectangle as the base of a prism with height a . Create the prism. What is the volume of this prism?

32. Compare the two prisms. Does $(a \cdot b) \cdot c = a \cdot (b \cdot a)$? Explain how you know this.

Distributive Property of Multiplication over Division:

The product of a number and the sum of two numbers is equal to the sum of the products of the first number with the second number and the first number with the third number. In symbols, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Note: We can use the Symmetric Property to rewrite the equation above as $a \cdot b + a \cdot c = a \cdot (b + c)$. This version of the Distributive Property is used to justify factoring out a common factor to write an expression in factored form.

33. Combine one line segment of length b and one line segment of length c to obtain a line segment of length $b + c$. Create a rectangle with length a and width $b + c$. Write an expression for the area of this rectangle (rectangle #1).

34. Create a rectangle with length a and width b . What is the area of this rectangle?

35. Create a rectangle with length a and width c . What is the area of this rectangle?

36. Combine the two rectangles to create one larger rectangle. Write an expression for the area of this new rectangle (rectangle #2).

37. Compare rectangle #1 and rectangle #2. What can you say about the two rectangles? What does this mean about the two expressions that show the area of each rectangle?

38. How could you use rectangles #1 and #2 to justify the statement $a \cdot b + a \cdot c = a \cdot (b + c)$?



Challenge Problem: Volume of a Pyramid

Determine the volume of a square pyramid.

1. Build the square pyramid shown on pages 8 and 9 of the Instructions Booklet.

In order to find the volume of the square pyramid, you will use the formula that you developed with the assistance of your teacher ($V = \frac{1}{3} Bh$).

- V = volume
- B = area of the base of the pyramid
- h = height of the pyramid

Use the appropriate letters to represent the line segments that make up the edges of the pyramid:

- s = silver line segment
- r = red line segment
- y = yellow line segment, etc.

2. What color rod represents the height of the pyramid?

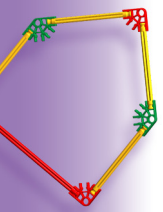
3. Find the volume of the square pyramid. Explain how you determined the volume.

4. Find the volume of one of the small pyramids that make up the large square pyramid that you constructed for this activity (the small pyramid is made from 8 yellow rods and 5 vertices). The volume of this smaller pyramid can be determined using more than one strategy. Explain how you determined the volume.

5. Construct the octahedron model on page 10 of the Instructions Booklet. Use information gathered while finding a solution to question 4 to determine the volume of the octahedron.

6. Construct the triangular pyramid model on page 10 of the Instructions Booklet. Can you develop a strategy to determine the volume of this polyhedron? (Hint: If you stand the model on one of the faces that forms an isosceles triangle, you will notice that the model is a section of a more common pyramid. This information should help you to develop a strategy to solve the challenge.) What is the volume?





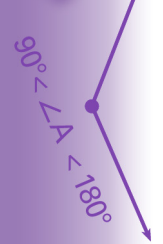
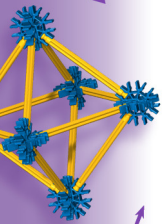
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$$y + s = s + y$$

$$y + y + w + w = 2y + 2w$$

$$w < r$$



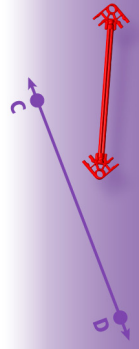
$$90^\circ < \angle A < 180^\circ$$



$$y = y$$



$$r + b = b + r$$



CD



LESSON 11

Addition of Polynomials

Lesson Topics: Defining and Adding Polynomials

Lesson Length: Two 50-minute periods

Student Objectives:

Students will:

- Understand the difference between constants and variables
- Know the terms; monomial, polynomial, numerical coefficient, degree, and descending order.
- Add polynomials.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX to create 1-D, 2-D, and 3-D representations of constants and variables. This will allow them to recognize like terms, so they know which terms can be combined when adding polynomials.
- Students will then look at the degree of a term, the degree of a polynomial, and descending order for a polynomial.

Materials and Equipment:

- K'NEX Middle School Math set and Instructions Booklets
- Copies of the Lesson #11 Student Inquiry Sheets

A – Motivation and Introduction:

1. Hold up a rectangular prism with dimensions b by r by s made with K'NEX (prepared in advance).

"The surface area for a prism such as this would be the sum of the areas of the six faces. We could represent the area of one face as br because its dimensions are a blue rod and a red rod. What are the areas of the other five faces?" "Could we simplify this sum of six areas?" "In the next investigation you learn how to simplify expressions such as this one."

2. Demonstrate examples of variable representations that include like terms because they are made of a combination of the same shape.

For example, refer to page 17 of the Instructions Booklet. The area of square $y + s$ can be described as $y^2 + 2ys + s^2$.

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Ask the students to simplify the expression for the surface area of the prism that was used to launch the activity.

Assessment:

Observe the students during the group work. Use a checklist to record whether students are using correct terminology and reasoning mathematically. Each group should receive a group grade on the activity. Ask the students to explain what they learned during the lesson in their Math Journal. Students should also record any concepts or ideas that are not clear to them at this time.

Extension:

A **Polynomials Challenge** that has been developed to explore students' understanding of concepts from this and earlier lessons and to test their mathematics reasoning and problem solving skills. This exercise is located on the final page of the Student Inquiry Sheets if you choose to use it with your students.

Teacher Notes: Challenge Problem - Polynomials

Students have learned a great deal about polynomials and the degree of the terms in polynomials. As students complete this challenge, they will have an opportunity to describe a variety of the K'NEX models outlined in the Instructions Booklet mathematically using polynomial notation. In addition to describing the models using polynomial terms, students will be able to describe the degree of the terms in the polynomial.

Operations on Polynomials: Addition, an Algebra Investigation

Representations:

Use different segment lengths to represent variables and constants. Make a copy of each of the constants and variables from K'NEX pieces. Transfer all 1-D and 2-D variables and constants to a sheet of paper (1) placing the model on the page, (2) using a crayon or pencil to place a dot in the center of each endpoint or vertex, (3) and after removing the model, connecting the endpoints with a ruler. Label each line segment with the appropriate value or letters. Use white connectors for the endpoints.

1-D Constants:

- 1 = One green rod and two connectors
- 2 = Two green rods and three connectors
- 3 = Three green rods and four connectors
- ... Etc.

1-D Variables:

- b = one blue rod with two connectors
- y = one yellow rod with two connectors
- r = one red rod with two connectors
- ... Etc.

Rectangles can also be used to represent constants and variables in two dimensions. The area of the rectangle tells us what the rectangle represents. We think of the rectangles as composed of multiple rectangles of the same dimensions. For example, a rectangle of 4 square units consists of four linked 1 unit squares, while $2br$ would be represented by two linked br rectangles. Use a pencil and ruler to transfer each of the rectangles to a sheet of paper.

2-D Constants:

- 1 square unit = 1 rectangle (square) that has an area of 1 (made with 4 green rods and 4 connectors)
- 2 square units = 1 rectangle that has an area of 2 (made with 6 green rods and 6 connectors)
- 4 square units = 1 rectangle that has an area of 4
- ... Etc.

2-D Variables:

- b^2 = a rectangle (square) with sides that are made of 4 blue rod segments
- y^2 = a rectangle (square) with sides that are made up 4 yellow rod segments
- br = a rectangle with sides that are made of 2 red rod segments and 2 blue rod segments
- ... Etc.

We can also use rectangular solids to represent constants and variables. The volume of the rectangular solid tells us what the solid represents. (Do not attempt to transfer these models to the back of your paper.)

3-D Constants:

- 1 cubic unit = 1 cube made of green rods and a number of blue and gray connectors
- 2 cubic units = 1 rectangular prism made from 16 green rods and a number of blue and gray connectors
- 3 cubic units = 1 rectangular prism made of more green rods, blue connectors, and dark gray connectors than are in the group's materials. Join with another group to form a rectangular prism that has the dimensions of $1 \cdot 1 \cdot 3$.
- ... Etc

3-D Variables:

- y^3 = 1 cube made of yellow rods and a number of blue and dark gray connectors
- b^2y = 1 rectangular prism whose base is a square made of blue rod segments and whose height is a yellow rod segment

Polynomials:

A **monomial** is a product of a real number and variables with whole number exponents. Thus, $2r$ and 3 as well as 7 and br are monomials.

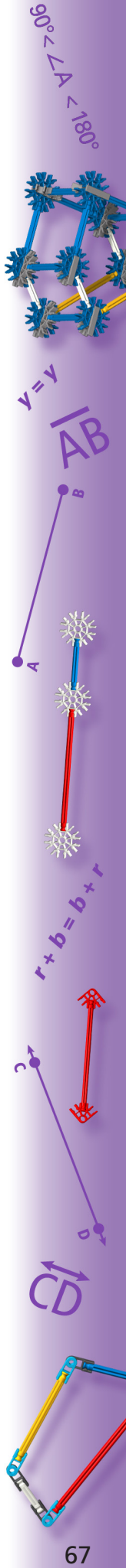
A **polynomial** is a monomial or the sum and/or difference of two or more monomials. Thus, $2y - 3$, $b + r$, $-3y^2 + r^2 - br$ and the monomials above are all polynomials.



Challenge Problem: Polynomials

During this polynomial lesson, you used a variety of terms and you determined the degree of these terms. The following challenges require that you utilize that information to describe the length, area, or volume of models that you can build following the instructions provided in your Instructions Booklet.

1. Describe the red line segment on page 3 of the Instructions Booklet with a single term. What is the degree of the term?
2. Build the models on page 4 of the Instructions Booklet and describe the perimeter of the models with as many terms as necessary. What is the degree of each term? What is the degree of the polynomial that describes each model?
3. Build the models on page 5 of the Instructions Booklet. Describe the number and color of the edges on the models using terms that include numbers and letters. What word(s) is used to describe the numbers that are a part of your terms? What is the degree of each term? What is the degree of the polynomial? You may also complete this activity for the models on pages 6-7, 8-9, 10, 14, 18 and 19 if time allows.
4. Build the right trapezoid model on page 4 of the Instructions Booklet. Describe the area of the right trapezoid as a polynomial. (Hint: Decompose the shape to form shapes that are easier to describe mathematically.) What is the degree of the polynomial?
5. Build the model on the right side of page 17 of the Instructions Booklet. Describe the area of the shape as a polynomial. What are the degrees of the terms in this polynomial?
5. Build the models on pages 18-21 of the Instructions Booklet. Describe the volume of the models as a polynomial. (Hint: Decompose the shape to form shapes that are easier to describe mathematically.) What is the degree of the polynomial term that describes each shape?



LESSON 12

Geometric Patterns

Lesson Topics: Arithmetic Sequences.

Lesson Length: Two 50-minute periods

Student Objectives:

Students will:

- Recognize an arithmetic sequence.
- Find the first term and common difference of an arithmetic sequence.
- Develop a formula for the n th term of an arithmetic sequence.
- Find a specified term of an arithmetic sequence.
- Determine the value of n when given the value of a term in an arithmetic sequence.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX materials to create various sequences of geometric patterns that exhibit arithmetic growth.
- Students will learn how to represent terms symbolically and how to find the common difference.
- They will then create a formula that can be used for finding the n th term in the sequence.
- Finally, they will use whatever method they choose to solve for n when they have been given the value of the n th term.

Materials and Equipment:

- K'NEX Middle School Math set and Instructions Booklets
- Copies of the Lesson #12 Student Inquiry Sheets
- Balances (are needed to complete the last part of the challenge activity at the end of the lesson.)

A – Motivation and Introduction:

"An apartment is rented by a couple. They pay \$1000 initially (\$500 security deposit plus \$500 rent for month 1) and \$500 per month for a year. A sequence showing the total amount they have paid during the year is: 1000, 1500, 2000, 2500, ..., 6500.

If they continue to rent the apartment at this monthly rent, how much would they have paid in total after 24 months? This is an example of an arithmetic sequence. In this lesson you will learn how to find a specified term in such a sequence."

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.

3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Ask students to solve the apartment rental problem that was used to introduce the activity.

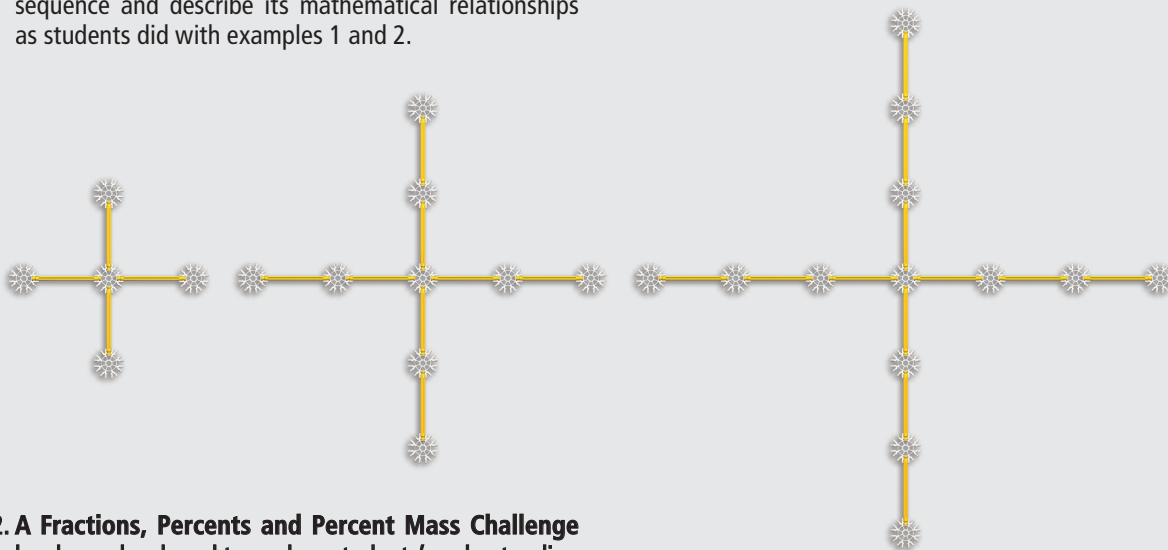
Assessment:

Observe the students during the group work. Use a checklist to record whether students understand the use of subscripts, what the common difference is, and the fact that the common difference is not added in obtaining the first term of the sequence. Each group should receive a group grade on the project. Ask the students to explain in their Math Journal what they learned during the lesson and any concepts that are still unclear.



Extension:

1. Determine whether this pattern forms an arithmetic sequence and describe its mathematical relationships as students did with examples 1 and 2.



2. A Fractions, Percents and Percent Mass Challenge has been developed to explore students' understanding of concepts and to test their mathematics reasoning and problem solving skills. This exercise is located on the final page of the Student Inquiry Sheets if you choose to use it with your students.

Teacher Notes:

Challenge Problem -

Fractions, Percents and Percent Mass

- During this challenge activity students will build models, organize data, and describe the make up of their models mathematically. The students will only need their K'NEX Middle School Math sets, pencil, and paper for the first two activities. To complete the Percent Mass activity, students will need a standard balance or an electronic balance. Balances that are accurate to $\frac{1}{10}$ of a gram or

better will provide excellent results as students complete the activity.

- Student designed models can also be used for this sequence of challenges. Students can then learn from both the mathematical design of their own model and the process of finding solutions to their classmates models.

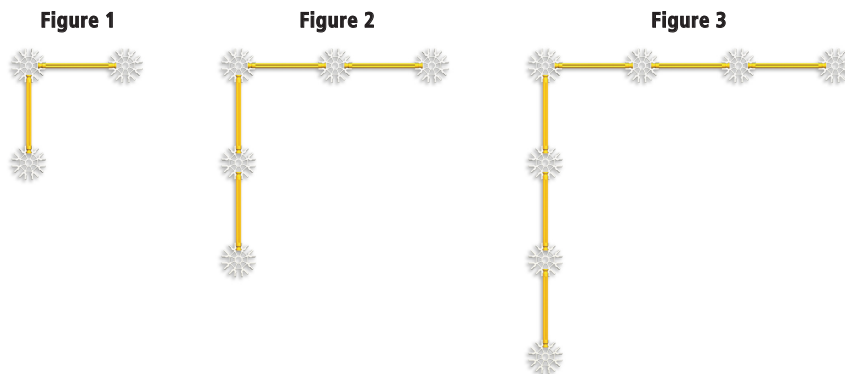


Geometric Patterns: Arithmetic Sequences

An arithmetic sequence is a sequence of numbers where the difference between consecutive numbers in the sequence is a constant. This constant difference is called the common difference. For example, if you have to put down \$1000 at the signing of the lease and pay \$299 per month to lease a car, the sequence showing the total amount paid for the lease after 1, 2, 3, etc. months is: 1299, 1598, 1897, ... The ellipses (...) mean "et cetera". The sequence 1299, 1598, 1897, ... is an arithmetic sequence with common difference 299.

Geometric Pattern #1:

Consider the following geometric pattern – Pattern #1. Create the first figure with two yellow, K’NEX line segments connected at right angles. In the second figure we added a pair of yellow segments. In the third figure another pair of yellow rod segments is added. Build these three shapes using K’NEX yellow rods and white connectors.



1. What patterns do you see in this sequence of figures? Describe as many patterns as you can find.
2. Following these three pattern #1 models create the next two shapes in this geometric pattern. Show figures 4 and 5 by drawing stick figure representations. The K’NEX models for figures 4 and 5 must be made with yellow rods and a combination of white and yellow connectors.
3. Look at your five shapes. Write a sequence showing the number of rods in figures 1, 2, 3, 4, 5, etc.
Rod sequence:
4. Is this an arithmetic sequence? Explain why.



We use a variable name consisting of a letter and subscript to name each of the terms in a sequence. If we use the letter r for our rod sequence, then $r_1 = 2$ is the first number in the sequence, $r_2 = 4$ is the second number in the sequence, r_5 is the fifth number in the sequence, and r_n is the n th number in the sequence. We will use the letter d to represent the common difference in an arithmetic sequence.

- Find the common difference d for the rod sequence that describes your series of models.

$$d =$$

Notice that

- $r_2 = r_1 + d$
- $r_3 = r_2 + d = (r_1 + d) + d = r_1 + 2d$
- $r_4 = r_3 + d = (r_1 + 2d) + d = r_1 + 3d$

- Write r_5 as a sum of r_1 plus a multiple of d .

$$r_5 =$$

- Now look at the previous number multiplied by d in the representations of r_2 through r_5 (1, 2, 3, 4, 5). What would you have to do to the subscript of r in each of the four cases above to get that number that is multiplied by d ?

This leads us to a general formula that can be used to find any term in our rod sequence:

$$r_n = r_1 + (n - 1)d$$

- Use this formula to find the number of rods in pattern 1 in figure 10. (Hint: $n = 10$.)

$$r_{10} =$$

- Suppose the number of rods in a figure in pattern #1 is 24. What figure number must this be? Show how you found the answer.

- Look at your five geometric shapes. Write a sequence showing the number of connectors in figures 1, 2, 3, 4, 5, etc.

Connector sequence:

- Is this an arithmetic sequence? Explain why.

If we use the letter c for our connector sequence, then $c_1 = 3$ is the first number in the sequence, $c_2 = 5$ is the second number in the sequence, c_5 is the fifth number in the sequence, and c_n is the n th number in the sequence.

- Find the common difference d for the connector sequence.

$$d =$$

Notice that

- $c_2 = c_1 + d$
- $c_3 = c_2 + d = (c_1 + d) + d = c_1 + 2d$
- $c_4 = c_3 + d = (c_1 + 2d) + d = c_1 + 3d$

- Write c_5 as a sum of c_1 plus a multiple of d .

$$c_5 =$$

- Look at the previous number multiplied by d in the representations of c_2 through c_5 (1, 2, 3, 4). What would you have to do to the subscript of c in each of the four cases above to get that number that is multiplied by d ?

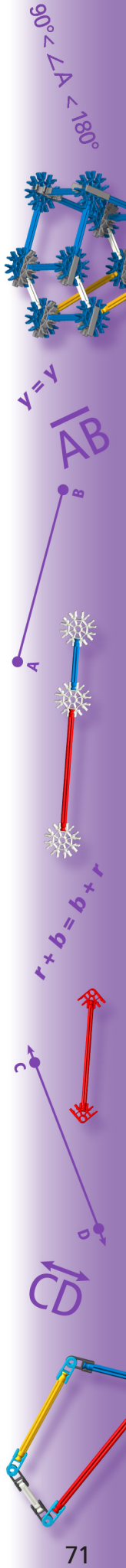
This leads us to a general formula that can be used to find any term in our connector sequence:

- $c_n = c_1 + (n - 1)d$

- Use this formula to find the number of connectors in pattern #1 in figure 10.

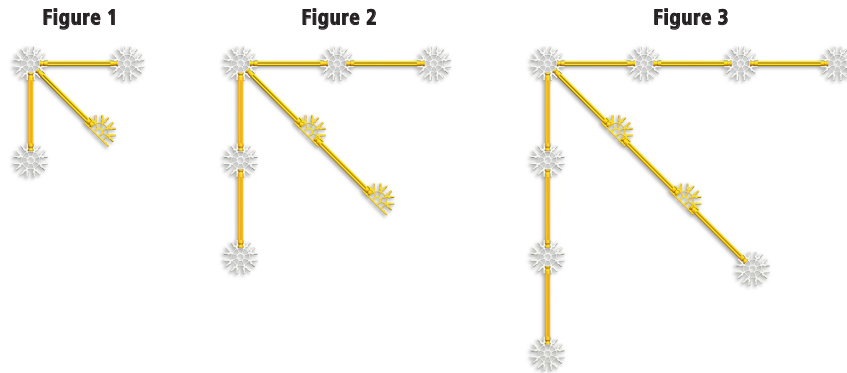
$$c_{10} =$$

- Suppose the number of connectors in a figure in pattern #1 is 35. What figure number must this be? Show how you found the answer.



Geometric Pattern #2

Consider the geometric pattern below. Create these three shapes using yellow rods and a combination of white and yellow connectors (there may not be enough white connectors depending on the number of students).



17. What patterns do you see in geometric pattern #2? Describe any that you see.

23. What would the number of rods be in figure 10 for pattern #2? Show how you found the answer.

18. Following these three pattern #2 models draw the next two shapes in this geometric pattern on a sheet of paper using stick figure representations.

24. If there are 36 rods in a figure for geometric pattern #2, what is the figure number? Explain how you found the number.

19. Write a sequence for the number of rods in figures 1, 2, 3, 4, 5, etc. for pattern #2.

Rod sequence:

25. Write a sequence for the number of connectors in figures 1, 2, 3, 4, 5, etc. for pattern #2.

Connector sequence:

20. Is this an arithmetic sequence? Explain why.

26. Is this an arithmetic sequence? Explain why.

21. What is the value of r_1 , the first number in the rod sequence of pattern #2?

27. What is the value of c_1 , the first number in the connector sequence of pattern #2?

22. What is the common difference, d , for the rod sequence?

$d =$

$c_1 =$



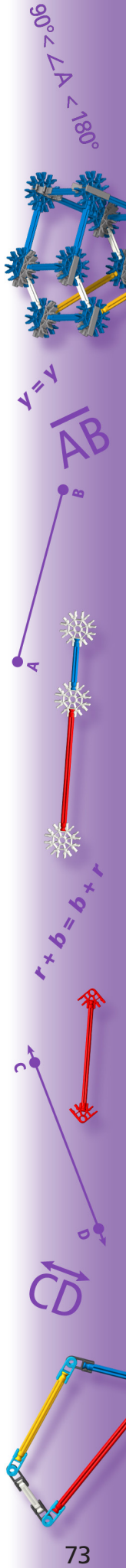
LESSON 12

28. What is the common difference, d , for the connector sequence?

$d =$

29. What would the number of connectors be in figure 10 for pattern #2? Show how you found the answer.

30. If there are 40 connectors in a figure for geometric pattern #2, what is the figure number? Explain how you found the number.



Challenge Problem: Fractions, Percents, and Percent Mass

The challenge is to describe the make up of several models you create using a variety of strategies and a variety of mathematical notation systems. Your teacher will provide direction as to which of the challenges you are to complete during this activity.

Build the models shown on page 11 and 12 of the Instructions Booklet.

Challenge #1:

1. Based on piece count, what fraction of the Trike model on page 11 in the Instructions Booklet is represented by each color K'NEX piece? (i.e., $\frac{1}{4}$ green, $\frac{1}{6}$ red, etc.) Gather and compute the same information for the Figure and Bridge models.

Challenge #2:

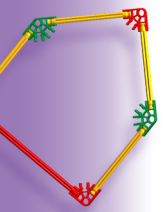
2. Based on piece count, what percent of the Trike model is represented by each color K'NEX piece? (i.e., 15% green, 25% red, etc.) (Please round numbers to the nearest percent.) Gather and compute the same information for the Figure and Bridge models.

Challenge #3:

For this activity, you will need to collect data using a balance.

3. Based on the mass of the various color pieces that make up the first model, what percent of the total mass of the model is represented by each color? (i.e., 10% green by mass, 30% red by mass, etc.) Gather the same information for the Figure and the Bridge models. Will your answers be more accurate if you find the mass of each piece and multiply to find the mass of the like pieces in the model or if you take the model apart and place, for example, all of the green pieces on the scale? Explain your response.





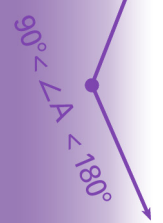
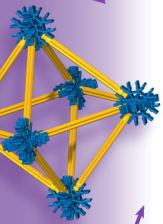
EF



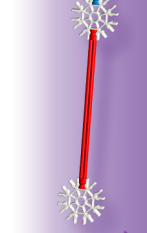
$$y + s = s + y$$

$$y + y + w + w = 2y + 2w$$

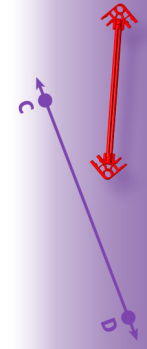
$$w < r$$



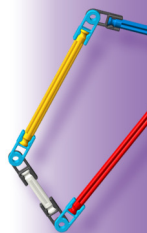
$y = y$
AB



$$r + b = b + r$$



CD



$90^\circ < \angle A < 180^\circ$

LESSON 13

Linear Patterns

Lesson Topics: Representations of Linear Functions.

Lesson Length: Three 50-minute periods

Student Objectives:

Students will:

- Recognize a linear relationship from a table or graph.
- Find the slope and y-intercept of a linear function from a table or graph.
- Find an equation for a linear function.
- Use a table, graph or equation to answer questions about a linear relationship.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX materials to create a train of right triangles and find the linear relationship between the number of rods and the number of triangles in the train.
- They will then find the linear relationship between the number of connectors and the number of triangles in the train. These linear functions will be represented as tables, graphs, and equations. The representations will be used to extend the patterns.
- The investigation is repeated with a train of squares.

Materials and Equipment:

- K'NEX Middle School Math set and Instructions Booklets
- Copies of the Lesson #13 Student Inquiry Sheets
- Graph paper
- Rulers

A – Motivation and Introduction:

"A book club charges \$20 to join the club. You then receive four books each month for \$15. A plumber charges \$50 for a service call plus \$65 per hour. These are just two of many examples of linear relationships. In this lesson we will explore linear patterns and learn different ways of representing them."

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Ask students to represent one of the problems from the introduction as a table, graph, and equation. Ask them to justify why each representation of a train of triangles

or squares is an accurate model of a linear pattern and to explain similarities between the models.

Assessment:

Observe the students during the group work. Use a checklist to record whether students can recognize a linear function from a table. Each group should receive a group grade on the project. Ask the students to explain in their Math Journal what they learned during the lesson and any concepts that are still unclear.

Extension:

Provide students with K'NEX models or charts, graphs or tables of data that may or may not represent linear patterns. Instruct students to demonstrate whether the information represents a linear pattern or not. If the pattern is linear, can the students determine the slope?



Linear Patterns: Representing Linear Functions

Train 1:

Create a right triangle using 2 blue rods, 1 yellow rod and 3 yellow or white connectors. (See figure 1.) We are going to create a train of these right triangles. Create the second figure which is formed using 4 blue rods and 1 yellow rod and connectors. Finally, add another triangle to the train to create figure 3.

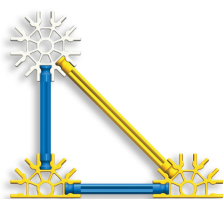


Figure 1

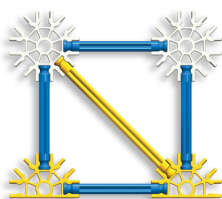


Figure 2

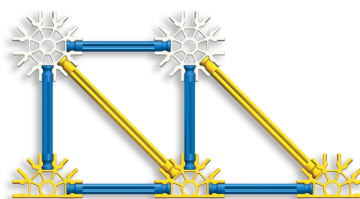


Figure 3

1. What patterns do you see in this train? Describe as many patterns as you can find.

first is 1 more than the previous one, the inputs have a common difference of 1 and this is an arithmetic sequence. Let the "number of rods" column represent the output values.

3. Is the number of rods an arithmetic sequence? If it is, what is the common difference?

1. Use these patterns to create the next two figures in the pattern. Sketch them on a sheet of paper using stick figures representations.

4. Is the number of rods a linear function of the figure number? Why?

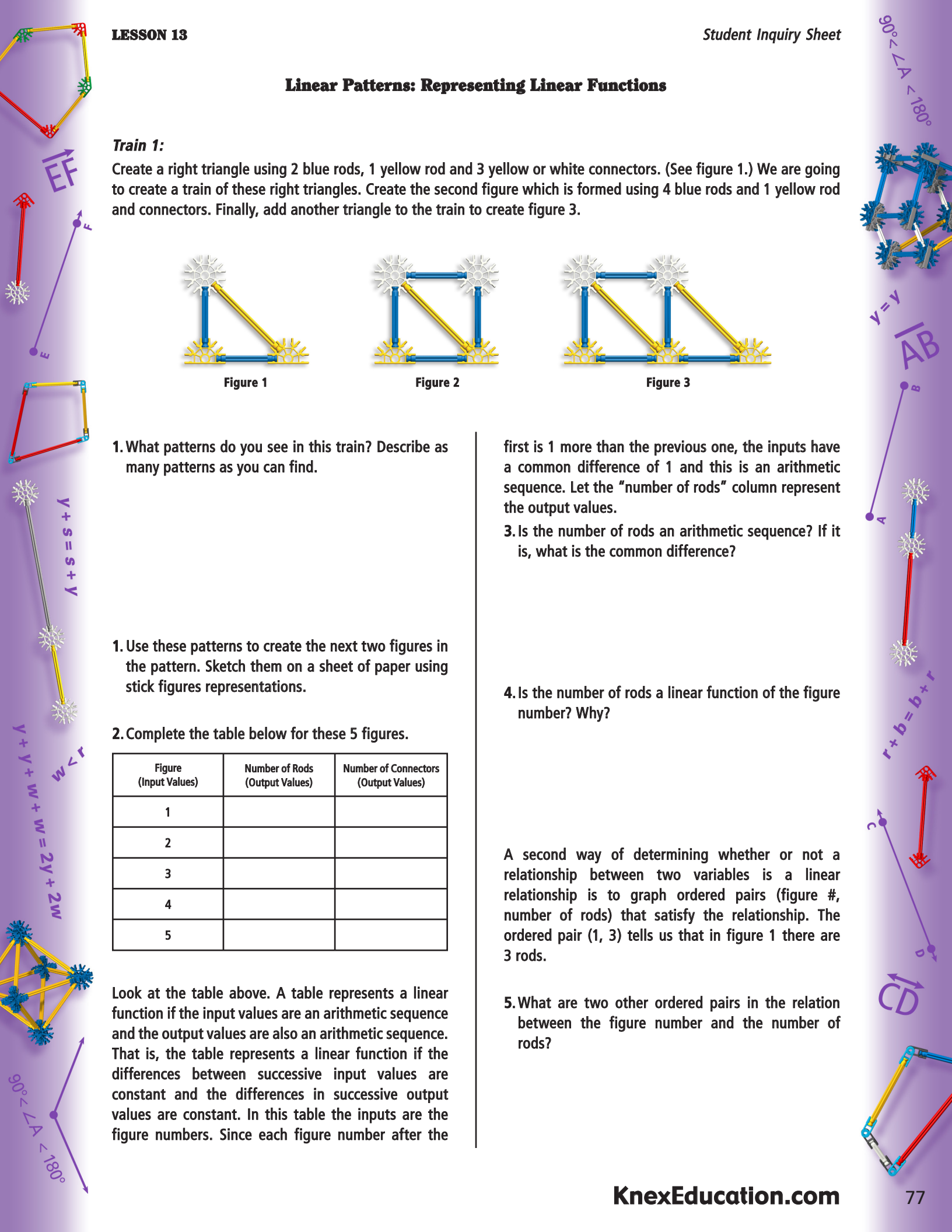
2. Complete the table below for these 5 figures.

Figure (Input Values)	Number of Rods (Output Values)	Number of Connectors (Output Values)
1		
2		
3		
4		
5		

A second way of determining whether or not a relationship between two variables is a linear relationship is to graph ordered pairs (figure #, number of rods) that satisfy the relationship. The ordered pair (1, 3) tells us that in figure 1 there are 3 rods.

5. What are two other ordered pairs in the relation between the figure number and the number of rods?

Look at the table above. A table represents a linear function if the input values are an arithmetic sequence and the output values are also an arithmetic sequence. That is, the table represents a linear function if the differences between successive input values are constant and the differences in successive output values are constant. In this table the inputs are the figure numbers. Since each figure number after the



LESSON 13

6. Plot the points corresponding to the ordered pairs (figure #, rods) in the table above on a sheet of graph paper. The horizontal axis (x) will represent the figure number. The vertical axis (y) will represent the number of rods.

7. Do the points on your graph lie in a straight line?

8. Draw a line through these points (use a ruler if you wish). Label the graph "Student Inquiry Sheet - Lesson #13 - Graph 1 - Triangle Rods".

A third representation of a linear function uses symbols. All linear functions can be represented as an equation in the form $y = mx + b$, where x is the input variable (here, the figure number) and y is the output variable (here, the number of rods). The number m is the ratio of the change in the y -values (the outputs) to the change in the x -values (the inputs). That is, it is the ratio of the common difference in the outputs (sometimes called the rise) over the common difference in the inputs (sometimes called the run). We call m the **slope** of the line, because it tells us **how steep the line is** and whether it is increasing or decreasing.

9. What is the slope of the line showing the linear relationship between the number of rods and the figure number?

$$m =$$

The number b is the output value that corresponds to an input of 0. Graphically, b is the point where the line crosses the vertical or y -axis. That is, b is the **y -intercept** of the graph of the linear function. If the value of the output when the input is 0 is given, it is easy to find b . If you cannot find b this way, you can put the value of m and the x -value and y -value of a point on the graph into the form $y = mx + b$ to obtain an equation with only one letter - b . Since the point (1, 3) is a point on our graph, we can replace x by 1 and y by 3.

10. Create an equation by doing these substitutions and substituting your value for m into $y = mx + b$. Then solve this equation for b . $y = mx + b$

$$b =$$

11. Use the values of m and b just found to represent the linear function showing how the number of rods needed is related to the figure number. You will write the formula for the slope of this linear function and substitute actual numbers for the ' m ' and the ' b ' in the formula.

$$y =$$

12. Use this equation to predict the number of rods needed for figure 10 of this pattern. (Remember, y is the number of rods needed and x is the figure number.)

13. Use this equation to find the figure number, if 25 rods are needed. Show all work.

14. Refer to the table on the first page. Is the number of connectors needed a linear function of the figure number. How do you know?

15. Plot the points corresponding to the ordered pairs (figure #, connectors) on the back of your other graph. The horizontal axis (x) will represent the figure number. The vertical axis (y) will represent the number of connectors.

16. Do the points lie in a straight line?

17. Draw a line through these points (a ruler will help). Label the graph "Student Inquiry Sheet - Lesson #13 - Graph 2 - Triangle Connectors".



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18. What is the slope of the line showing the linear relationship between the number of connectors and the figure number?

$m =$

19. What is the y-intercept b for this linear function?

$b =$

20. Write an equation that represents the linear relationship between the number of connectors needed and the figure number for this train.

$y =$

21. Use this equation to predict the number of connectors needed for figure 12.

22. If 20 connectors are needed for a figure in this train, what is the figure number? Show how you found the answer.



Train 2:

This train is formed by connecting squares together. Create the first three figures shown below using yellow rods and yellow and white connectors.

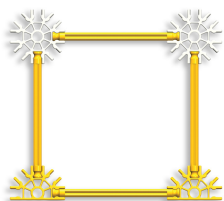


Figure 1

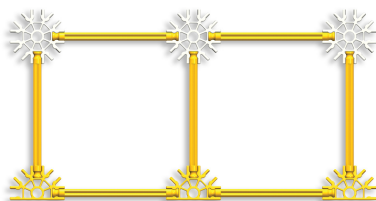


Figure 2

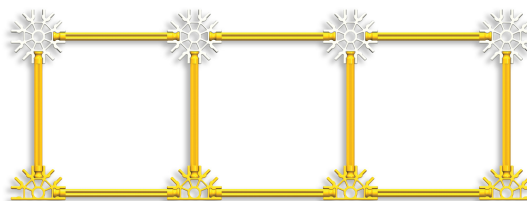


Figure 3

1. What patterns do you see in this train? Describe as many patterns as you can find.

2. Use these patterns to create the next two figures in the pattern. Sketch them on a sheet of paper using stick figure representations.

3. Complete the table below for these 5 figures.

Figure (Input Values)	Number of Rods (Output Values)	Number of Connectors (Output Values)
1		
2		
3		
4		
5		

4. Is the number of rods a linear function of the figure number? How do you know?

5. Use the table to determine how many rods will be needed for figure 6 if you were to build that figure from K'NEX.

6. Plot the points corresponding to the ordered pairs (figure #, rods) found in the table on a sheet of graph paper. The horizontal axis (x) will represent the figure number. The vertical axis (y) will represent the number of rods. Draw a line through these points (a ruler will help). Label the graph "Student Inquiry Sheet - Lesson #13 - Graph 3 - Square Rods".

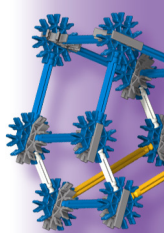
7. Use the graph to predict how many rods will be needed in figure 8. How did you find the answer?

8. Find the slope m and the y -intercept b of this linear function.

$m =$

$b =$

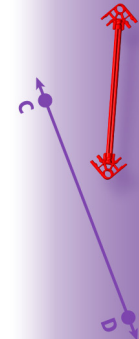
$90^\circ < \angle A < 180^\circ$



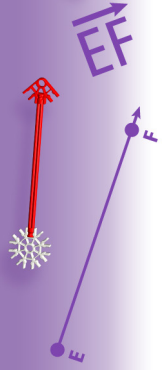
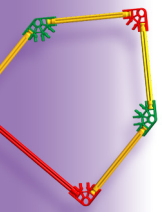
$y = y$
 \overline{AB}



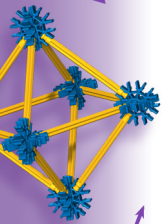
$r + b = b + r$



\overline{CD}



$y + s = s + y$
 $w < r$
 $y + y + w + w = 2y + 2w$



$90^\circ < \angle A < 180^\circ$

LESSON 13

9. Write the equation that represents this linear function.

$y =$

10. Use this equation to predict how many rods will be needed for the figure 10 train. Show all your work.

11. If 37 rods are needed to create a train in this sequence, what is the figure number? How did you find the answer?

12. Refer to the table for train 2. Is the number of connectors needed a linear function of the figure number? Explain how you know.

13. Use the table to predict how many connectors will be needed for figure 6.

14. Use the table to predict how many connectors will be needed for figure 7.

15. Refer to the table for train 2. Plot the points corresponding to the ordered pairs (figure #, connectors) found in previous table on the back of your other graph. The horizontal axis (x) will represent the figure number. The vertical axis (y) will represent the number of connectors. Draw a line through these points (a ruler will help). Label the graph "Student Inquiry Sheet - Lesson #15 - Graph 4 - Square Connectors".

16. Use the line on your graph to predict how many connectors will be needed in figure 8.

17. Find the slope m and the y -intercept b of this linear function.

$m =$ $b =$

18. Write the equation that represents this linear function.

$y =$

19. Use this equation to predict how many connectors will be needed for figure 10 of the train. Show all your work.

20. If 26 connectors are needed to create a train in this sequence, what is the figure number? How did you find the answer?



LESSON 14

Fractal Trees

Lesson Topics: Arithmetic and Geometric Sequences.

Lesson Length: Two 50-minute periods

Student Objectives:

Students will:

- Discover that arithmetic sequences are additive while geometric sequences are multiplicative.
- Understand the definition of a fractal figure.
- Extend a geometric sequence.
- Define a recursive function for a sequence.
- Discover that arithmetic sequences are linear and geometric sequences are nonlinear.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX to create a fractal tree and discover that the number of branches added at each step (iteration) is a geometric sequence.
- They will investigate the differences between arithmetic and geometric sequences.
- They will discover that arithmetic sequences are linear functions while geometric sequences are nonlinear.

Materials and Equipment:

- K'NEX Middle School Math set and Instructions Booklets
- Copies of the Lesson #14 Student Inquiry Sheets
- Graph paper
- Large sheets of drawing paper
- Rulers

A – Motivation and Introduction:

1. A floret of broccoli can be used to introduce the idea of a fractal figure. The students can see how the floret is similar to the whole – self-similarity. The teacher can also talk about how fractal geometry has been used both to describe natural objects as well as its usage in computer graphics and encoding.
2. The students will use K'NEX to create fractal trees and explore some mathematics derived from the trees.

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Give examples of arithmetic and geometric sequences

and ask students to decide which is which. You can use white boards or flash cards to gather formative assessment feedback as to whether or not they recognize the two types of sequences they worked with during this activity.

Ask students which sequence is additive and which is multiplicative. Give them several graphs and ask whether each one represents an arithmetic sequence (the points lie in a straight line) or a geometric sequence (the points lie on a "J" curve).

Assessment:

Observe the students during the group work. Use a checklist to record the level of student's understanding. Each group should receive a group grade on the project. Ask the students to explain in their Math Journal what they learned during the lesson and any concepts that are still unclear.

Extension:

Students can use a calculator to find how many branches (rods) would be needed for later iterations. This can lead to a discussion of scientific notation and how scientific notation appears on a calculator.

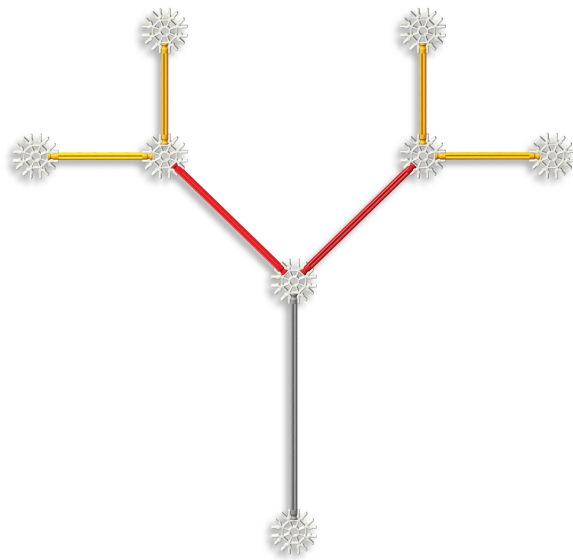


Fractal Trees: Arithmetic and Geometric Sequences

Fractal 1:

Construct a fractal tree with the assistance of another group in the classroom. Each of the two groups will build one half of the fractal tree. To create a fractal tree we start with the trunk – usually a line segment. Form the trunk of a fractal tree by creating a line segment using a silver rod and two white connectors. (See the left side of page 26 of the Instructions Booklet.)

The remainder of the fractal tree is formed in stages called **iterations**. The trunk of our tree has a base connector and a second connector. We will call any connector (except the base) that has only one rod connected to it a **leaf**. Thus the bottom white connector on the silver rod is the base and the top white connector is a leaf. When forming each iteration of the fractal tree add the same number of segments (**branches**) at the same angles to each leaf. The rods that one group will add to the right side of the tree in iterations 1 and 2 are shown in the center of page 26. The second group will add a mirror image of these pieces to the left side of the tree. Both groups will then add leaves to their yellow rods to form the image shown below (Iteration 2).



Iteration 2

Each group will continue to add iterations to the fractal tree using segments that get smaller by a constant ratio during each iteration (blue, white and green rods). Add two branches at a 45° angle to each leaf on the fractal tree to form the next iteration. Iterations 3, 4 and 5 for the group adding pieces to the right side of the model are shown on the right side of

page 26 in the Instructions Booklet. The other group will add a mirror image of the same pieces to the left side of the tree.

one another. (After adding the green rods, place a purple connector on the endpoint of each of those green rods. This will allow you to transfer your fractal to paper more easily later in this lesson.)

Theoretically, we could continue to add branches and leaves to the fractal forever. One of the properties of a fractal figure is that it is **self-similar**. That is, if you look at one part of the figure it will be similar to the whole figure or another, larger portion of the figure.

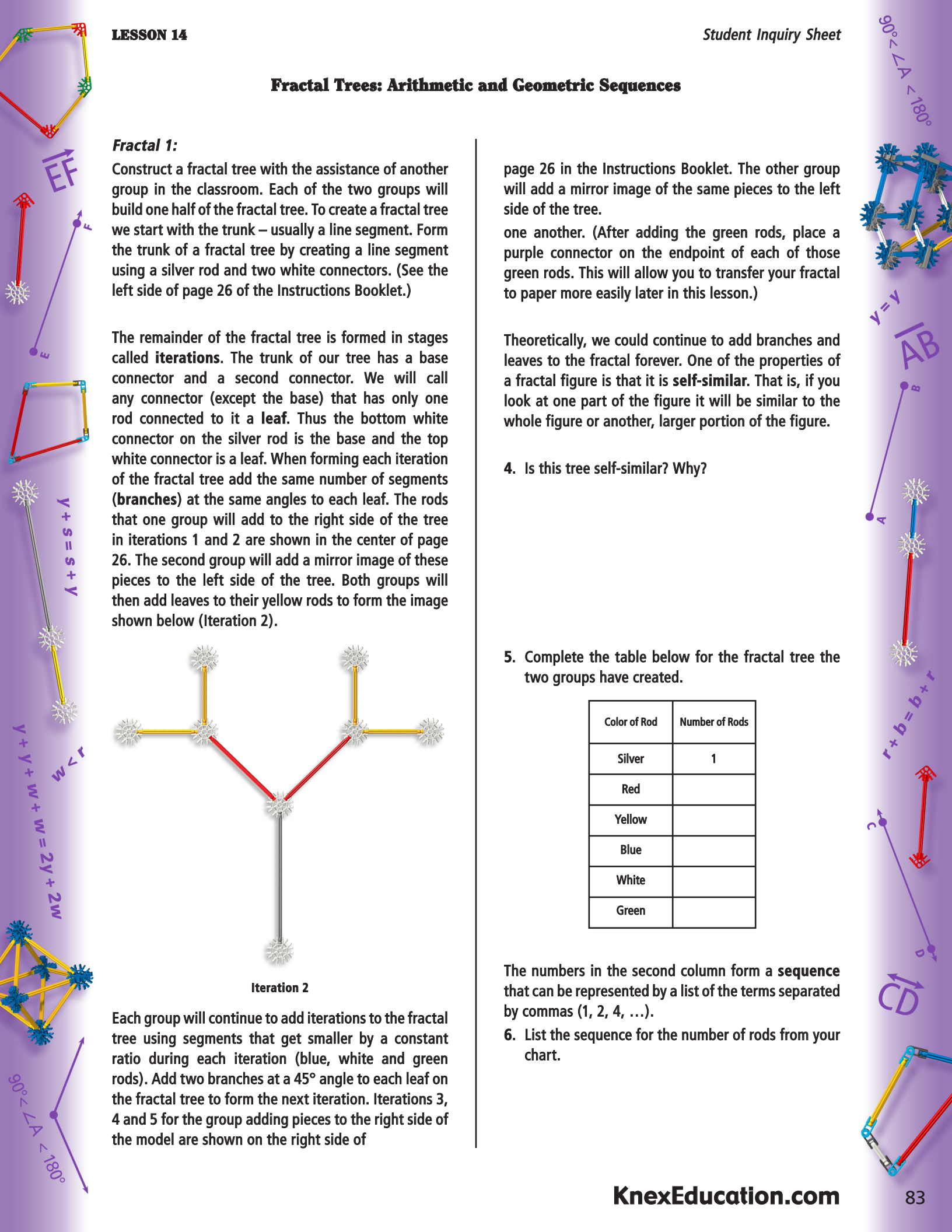
4. Is this tree self-similar? Why?

5. Complete the table below for the fractal tree the two groups have created.

Color of Rod	Number of Rods
Silver	1
Red	
Yellow	
Blue	
White	
Green	

The numbers in the second column form a **sequence** that can be represented by a list of the terms separated by commas (1, 2, 4, ...).

6. List the sequence for the number of rods from your chart.



Many problem situations can be modeled by either an **arithmetic sequence** or a geometric sequence. An arithmetic sequence is a sequence where the next term in the sequence can be obtained from the previous term by adding a constant called the **common difference** of the sequence. For example, suppose you were given a box of 30 chocolates for Valentine's Day and you eat 2 each day after Valentine's Day until they are gone. This could be represented by the arithmetic sequence: 30, 28, 26, 24, ..., 0.

Note that $28 - 30 = 26 - 28 = -2$, so the common difference is -2 . To get the number in the sequence after 24 we add -2 . The next number is $24 + (-2) = 22$. A sequence is arithmetic if the difference of a number in the sequence (after the first term) and the previous number is always the same number, the common difference.

7. Is the sequence based on the number of rods from your chart an arithmetic sequence? Justify your answer.

A **geometric sequence** is a sequence where the next term in the sequence can be obtained from the previous term by multiplying by a constant called the **common ratio**. For example, suppose a community health organization identified 5 cases of flu on the first day of an outbreak and the number of cases doubles each day. We can represent this situation by the sequence 5, 10, 20, 40,

Note that this sequence is not an arithmetic sequence, since $10 - 5 = 5$, while $20 - 10 = 10$, so the differences are not constant.

However, we can obtain the next number in the sequence by multiplying the current term by 2. We can determine if a sequence is geometric by dividing several terms (after the first term) by the previous term. If these ratios are all the same, the sequence is a geometric sequence.

Note that $10 \div 5 = 20 \div 10 = 2$.

9. Is the sequence based on the number of rods from your chart a geometric sequence? Justify your answer.

Notation

In order to describe the pattern to the number of branches added at each iteration, we need to define some notation. We let a_1 represent the first term of a sequence, a_2 the second term of the sequence, a_3 the third term of the sequence, etc. Thus, for the sequence 5, 10, 20, 40, ... $a_1 = 5, a_2 = 10$, etc.

We let d represent the common difference of an arithmetic sequence and r represent the common ratio of a geometric sequence. We use n to represent a term number. If $n = 3$, then $a_n = a_3 = 10$ for the latest sequence.

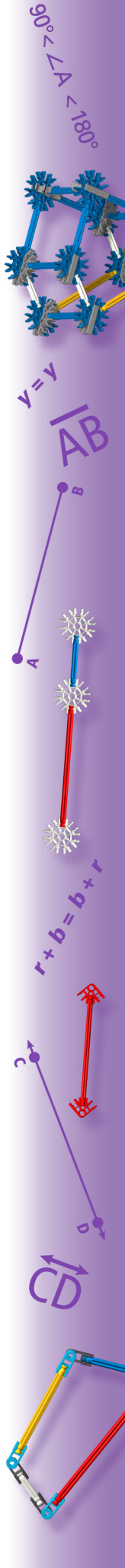
We can use a **recursive definition** to find the n th term of an arithmetic or geometric sequence. For arithmetic sequences we find the next term by adding the common difference to the current term: **Next Term = Current Term + Common Difference**.

Using our notation we can find the n th term using the recursive formula: $a_n = a_{n-1} + d$.

For a geometric sequence the corresponding formula is: $a_n = a_{n-1} \cdot r$.

Recursive formulas are used when you want to generate a sequence using a spreadsheet. For example, the recursive formula for the sequence 5, 10, 20, 40, ... is $a_n = a_{n-1} \cdot 2$.

10. Find a recursive formula for the Valentine's chocolates sequence 30, 28, 26, 24,



LESSON 14

We can also write a closed formula for the nth term of a sequence. Consider the arithmetic sequence: 30, 28, 26, 24, ...

$$\begin{aligned}
 a_1 &= 30 \\
 a_2 &= 30 + (-2) \\
 a_3 &= 30 + (-2) + (-2) \\
 a_4 &= 30 + (-2) + (-2) + (-2) \\
 &\text{Etc.}
 \end{aligned}$$

11. What pattern(s) do you see to this representation of the terms of the arithmetic sequence? Describe any patterns you find.

Consider the geometric sequence: 5, 10, 20, 40, For this sequence,

$$\begin{aligned}
 a_1 &= 5 \\
 a_2 &= 5 \cdot 2 \\
 a_3 &= 5 \cdot 2 \cdot 2 \\
 a_4 &= 5 \cdot 2 \cdot 2 \cdot 2 \\
 &\text{Etc.}
 \end{aligned}$$

12. What pattern(s) do you see to this representation of the terms of the geometric sequence? Describe any patterns you find.

Consider the sequence for the number of branches added to our fractal tree at iterations 1, 2, 3, ... :

$$2, 4, 8, \dots$$

13. What type of sequence is this, arithmetic or geometric?

14. What is the value of a_1 for the sequence: 2, 4, 8, ...?

$$a_1 =$$

15. Find d or r , whichever is appropriate, for this sequence.

16. Find the 12th term of this sequence using the appropriate formula.

We can represent a sequence using a table or graph as well as numerically. Below are partial tables for the arithmetic sequence 30, 28, 26, 24, ... and the geometric sequence 2, 4, 8, ...

17. How can you tell from a table if the output values form an arithmetic or geometric sequence?

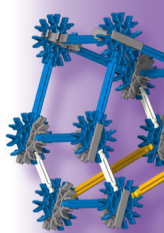
Valentine Chocolates

Day	# Left
1	30
2	28
3	26
4	24

Fractal Tree Branches

Iteration #	Leaves
1	2
2	4
3	8
4	16

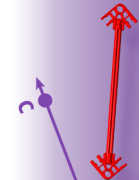
$90^\circ < \angle A < 180^\circ$



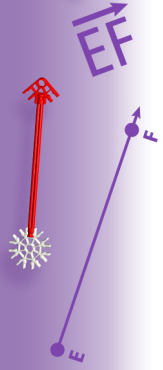
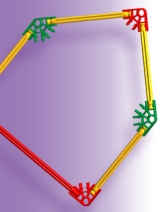
$v = v$
 \overline{AB}



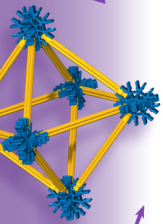
$r + b = b + r$



\overline{CD}



$v + s = s + v$
 $w < r$
 $v + v + w + w = 2v + 2w$



$90^\circ < \angle A < 180^\circ$

LESSON 14

When each of the groups are done and there are two large completed fractal trees, both will be combined to form an even larger fractal. Remove the white base connector from the two trucks and connect them together with a straight orange connector. This fractal will be used to answer the following questions.

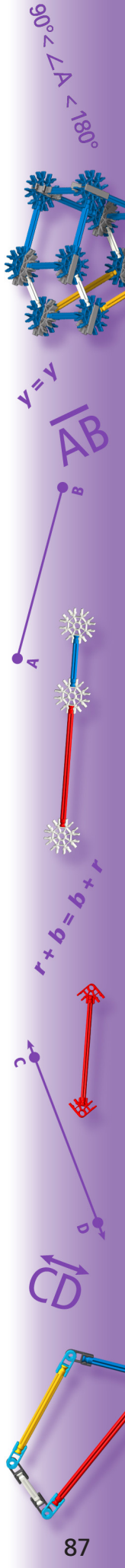
20. Create a sequence for the number of branches added at iterations 1, 2, 3, etc. for the model made when all four groups combined their sections to complete the fractal. State the type of sequence formed.

21. Find a recursive formula for the number of branches of this model.

22. Create a table for the number of branches added as a function of the iteration number.

23. Create a graph of this function on a sheet of graph paper.

Each group will transfer the fractal to large sheets of paper using a pencil and a ruler. Each group can then complete the remainder of the page with information discovered and learned about this model (data chart, graph, formula, etc.).



LESSON 15

Fractal Trees - Exponentials

Lesson Topics: Adding fractions, subtracting fractions, and developing rules.

Lesson Length: Two 50-minute periods

Student Objectives:

Students will:

- Be able to measure lengths to the nearest tenth of a centimeter.
- Be able to create a scatter plot of data from a data table.
- Find the equation of an exponential model from a table of data.
- Know the difference between exponential growth and decay.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Students will use K'NEX rods and connectors to create a fractal tree that uses smaller and smaller rods at each iteration.
- They will measure the length of each segment to the nearest tenth of a centimeter.
- The (iteration #, length) ordered pairs will be graphed to form a scatter plot that exhibits exponential decay.
- The ratios of consecutive lengths will be used to find the common ratio b in the exponential function model $y = a \cdot b^x$ for this graph.
- They will find the function model in symbols and use this equation and a calculator to answer questions about the fractal tree.

Materials and Equipment:

- K'NEX Middle School Math set and Instructions Booklets
- Copies of the Lesson #15 Student Inquiry Sheets
- Calculators
- Metric rulers
- Graph paper
- Large sheets of drawing paper

A – Motivation and Introduction:

1. "Many phenomena can be modeled using an exponential model. For example, the number of people who get the flu during the early days of an epidemic grows exponentially and the amount of radioactive material in a ton of waste from a nuclear plant decays exponentially."

"In the next lesson we will learn to recognize an exponential function from a table, graph, or equation and create an exponential model for the lengths of the segments in a fractal tree."

2. Demonstrate how to use a ruler to measure the length of a segment created using a rod and two connectors by measuring from the center of one circle in the connector on one end to the center of the circle in the connector on the opposite end.

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.

2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.

C – Summary and Closure:

Give each group a different set of data. Some sets will be linear and some exponential. Ask the groups to determine if their data set represents a linear or an exponential function. Groups will report their findings and justify their answer.

Assessment:

Observe the students during the group work. Use a checklist to record whether students understand the use of scatter plot and the exponential function model. Each group should receive a group grade on the project. Ask the students to explain in their Math Journal what they learned during the lesson and any concepts that are still unclear.



Extension:

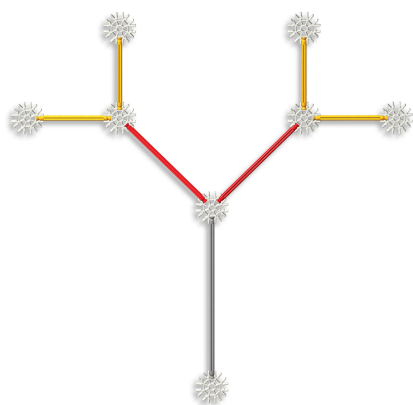
Create a spiral of triangles using K'NEX by starting with a right isosceles triangle with legs of length 1 (green rod segment). At each iteration add a larger right isosceles triangle with one leg the hypotenuse of the previous right triangle. Write a sequence that shows the length of the legs of the right triangle at each iteration. What type of function is represented by the (iteration, leg length) pairs?



Fractal Trees - Exponential Function Models

Note: see Lesson #14 - Fractal Trees: Arithmetic and Geometric Sequences for the definitions related to fractal trees.

Two groups will work together to form a fractal tree based on the pattern shown below. This is the same fractal tree that was constructed for Lesson #14. The final rods added to the fractal will be green. (After adding the green rods, place a purple connector on the endpoint of each of those green rods. This will allow you to transfer your fractal to paper more easily.)



Iteration 2

Your task is to find a formula for the length of a branch added at the n th iteration for this fractal. To find the solution you must measure the length of the segments created with different rods in successive iterations. Measure each segment to the nearest tenth of a centimeter. To find the length of a segment we measure the distance between endpoints on the connectors at each end as shown below. Transferring the various segments to paper will make measurements easier.



1. Fill in the table below using your measurements.

Iteration Number	Color of Rod	Segment Length (cm)
0	Silver	
1	Red	
2	Yellow	
3	Blue	
4	White	
5	Green	

2. On a separate sheet of graph paper, plot a point for each ordered pair (iteration number, segment length). The X-axis of the graph will include the iteration numbers and the Y-axis will include the segment lengths. Connect the points with a smooth curve.

The graph you created is a graph of an **exponential function**. Because the y -values get smaller as you move to the right, this is an example of what is called **exponential decay**. If you have a similar curve where the y -values get larger as you move to the right, we have what is called **exponential growth**. All exponential functions can be represented by an equation in the form $y = a \cdot b^x$. Note that the input variable x is an exponent; this is where the name for the function originated. The number a is the **initial value** of the function, the function or y -value when $x = 0$.

3. What is the value of a for this function?

$a =$

4. Where is the point corresponding to a located on the graph?

The number b in $y = a \cdot b^x$ is called the **base**. The number b is a unit rate of change in the y -values of the function. To find b we can find two y -values that correspond to x -values that are 1 apart and divide the second one by the first one. In the table on page 2 all of the input or x -values differ by 1, so we could use any two consecutive y -values.

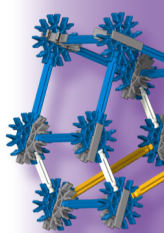
$b = \frac{\text{Length of red segment}}{\text{Length of gray segment}} = \frac{\text{Length of yellow segment}}{\text{Length of red segment}} = \dots$ etc.

Divide the length of the red segment by the length of the silver segment and round your answer to the nearest hundredth. Repeat this for the lengths of the yellow and red segments.

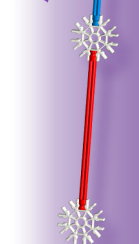
5. Are they close to the same number? Use the average of these two ratios as the value of b .

$b =$

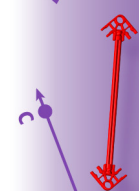
$90^\circ < \angle A < 180^\circ$



$y = y$
 \overline{AB}



$r + b = b + r$



c



\overline{CD}



LESSON 16

Sierpinski's Triangle

Lesson Topics: Fractal geometry; Exponential Growth and Decay

Lesson Length: One 50-minute periods

Student Objectives:

Students will:

- Find patterns in Sierpinski's triangle.
- Recognize that one pattern is exponential growth and another is exponential decay.
- Represent the patterns as tables, graphs and equations.
- Use the representations to answer questions concerning Sierpinski's triangle.

Grouping for Instruction:

- Whole group for launch and closure.
- Small groups for the investigation.

Overview of Lesson:

- Prior to the beginning of the lesson, please make one copy of the triangle shown on page 23 of the Instructions Booklet. When students attempt to make iteration 3 of their large Sierpinski's triangle they will need one additional copy of that triangle.
- Students will first work in a small group to create the first two iterations of the triangle using K'NEX rods and connectors. They will then combine their triangles with the triangles of other groups and your triangle to create the next iteration of the triangle.
- The groups can continue to combine the triangles to create other iterations of the triangle, if you have additional rods and connectors.
- Students will then use this large triangle to create a table for the number of triangles left after iterations 0, 1, 2, 3, and 4. The values in the table will be used to create a graphical representation of the number of triangles left. They will use the table values to find a common ratio b . The initial value a and common ratio b will be used to create a symbolic representation of this exponential function in the form $y = a \cdot b^x$. The relationship between the area of a single triangle left in Sierpinski's triangle and the iteration number will then be represented using a table, graph, and equation.
- Students will use the various representations to predict what would happen with later iterations of the triangle.

Materials and Equipment:

- K'NEX Middle School Math set and Instructions Booklets
- Copies of the Lesson #16 Student Inquiry Sheets
- One extra (teacher built) Sierpinski's triangle model as shown on the left side of page 23 of the Instructions Booklet (replace the bottom left yellow connector of your model with a red connector)

Note: You will need a large area in which to lay out Sierpinski's triangle

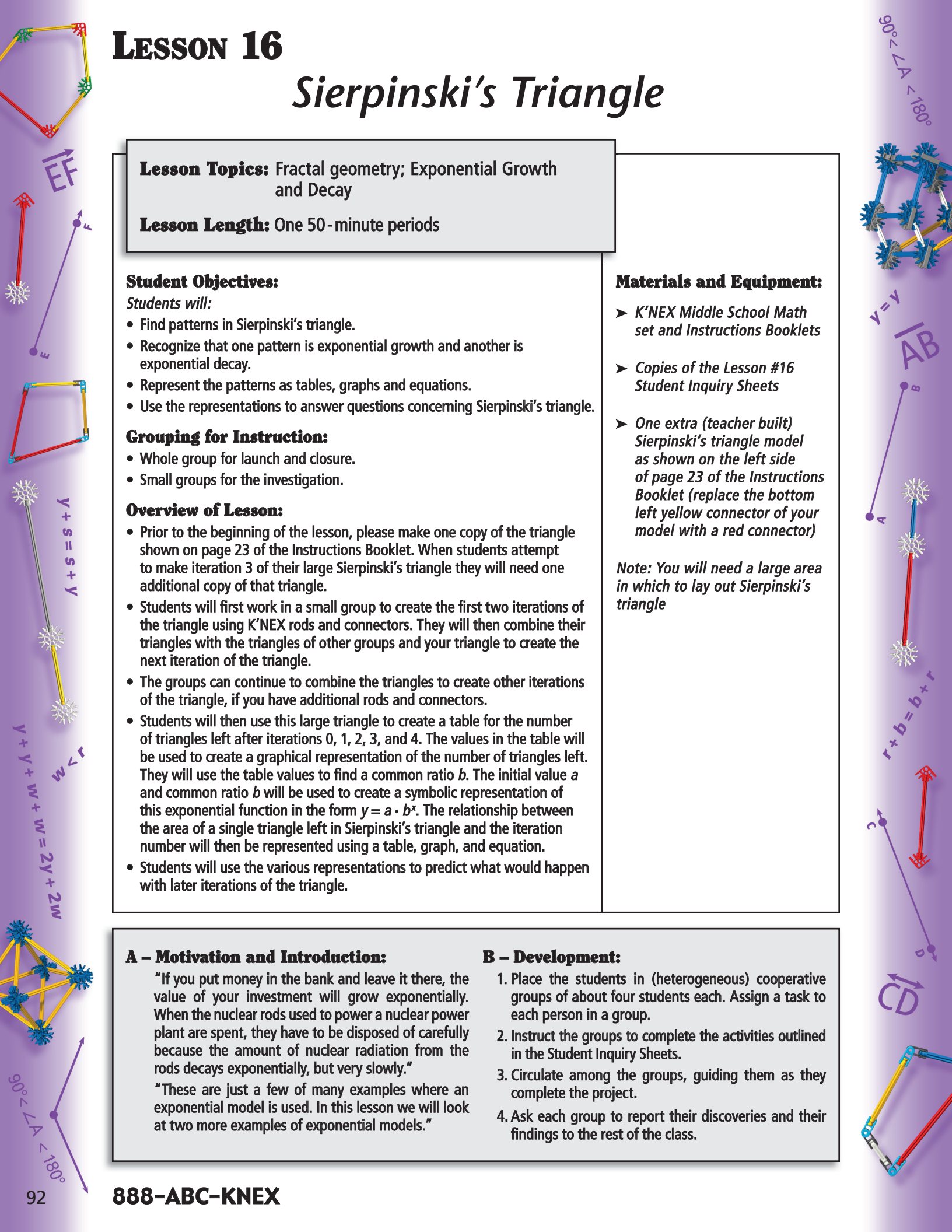
A – Motivation and Introduction:

"If you put money in the bank and leave it there, the value of your investment will grow exponentially. When the nuclear rods used to power a nuclear power plant are spent, they have to be disposed of carefully because the amount of nuclear radiation from the rods decays exponentially, but very slowly."

"These are just a few of many examples where an exponential model is used. In this lesson we will look at two more examples of exponential models."

B – Development:

1. Place the students in (heterogeneous) cooperative groups of about four students each. Assign a task to each person in a group.
2. Instruct the groups to complete the activities outlined in the Student Inquiry Sheets.
3. Circulate among the groups, guiding them as they complete the project.
4. Ask each group to report their discoveries and their findings to the rest of the class.



C – Summary and Closure:

Present a table of a linear function and an exponential function and ask which is exponential. Show an example of a parabola and an exponential "J" curve and ask students to identify the exponential function. Finally, ask students which of the equations $y = 1.2x^2$ and $y = 1.2 \cdot 3^2$ represents an exponential function.

Assessment:

Observe the students during the group work. Use a checklist to record whether students understand the use of ratios, tables, graphs, and exponential equations. Each group should receive a group grade on the project. Ask the students to explain in their Math Journal what they learned during the lesson and any concepts that are still unclear.

Extension:

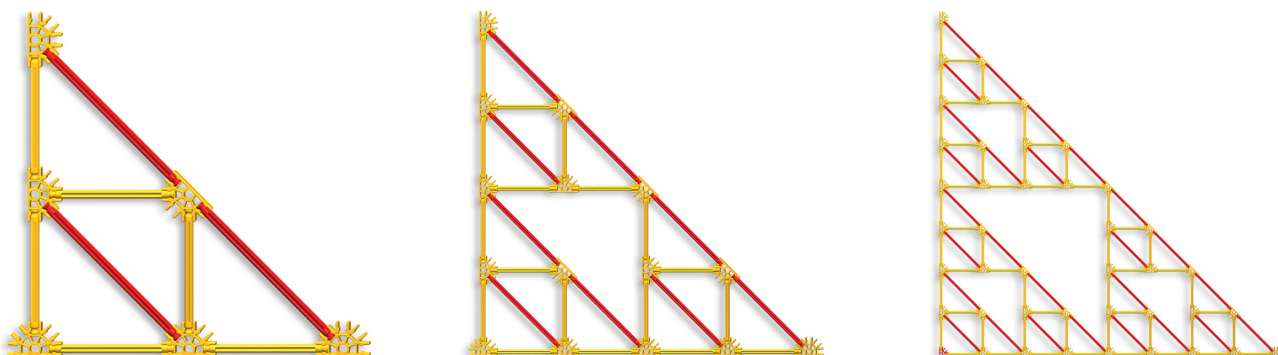
Students are asked to find an example of a real-life situation that illustrates exponential growth or decay. Students can be challenged to attach several strips of paper from a large roll of paper together so that they can transfer an iteration 2 or iteration 3 Sierpinski's triangle to paper. After the shape has been drawn, the students should shade in all of the triangles with a different orientation that are not considered a part of a Sierpinski's triangle. A label and appropriate mathematics explanations will make this diagram an excellent poster or wall mural.



Sierpinski's Triangle

Sierpinski's triangles can be formed using K'NEX materials in the following way.

- Start with a triangle. Use two yellow rods, a red rod and three yellow connectors to create a right triangle. The connectors represent the vertices A, B, and C. This is iteration 0 in the figures below.
- Create iteration 1 as shown below by referring to page 23 in the Instructions Booklet. We imagine that the interior triangle, the one with a different orientation when compared to the other three, is not part of the figure.
- Create a second copy of iteration 1 so that you may combine triangles with other groups in the classroom to form iteration 2 (shown below) of the Sierpinski's Triangle. Once again, the triangles with a different orientation are not part of Sierpinski's triangle.



Work with other groups and use a model prepared by your teacher to connect three different iteration 2 triangles to create a still larger triangle (you may need to lay this model out on an open section of the floor). This is iteration 3 of Sierpinski's Triangle.

1. Sketch iteration 3 in the space below and shade in the triangles that are not part of Sierpinski's triangle.

3. Use the figures and information for iterations 0 through 4 to complete the table below.

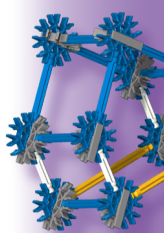
Iteration Number	Number of Small Triangles	Ratio of Outputs
0	1	1
1	3	$\frac{3}{1} = 3$
2		
3		
4		

If the relationship between the number of small triangles and the iteration numbers is a linear function, the differences of consecutive outputs would all be the same. Clearly, this is not the case. If the relationship between the number of small triangles and the iteration numbers is an exponential function, the ratio of consecutive outputs would be the same. The first ratio has been done for you.

2. If you had enough pieces to make iteration 4 of this triangle, how many triangles in the model would be shaded in and how many would be left open?

4. Find the remaining ratios by dividing the current output value by the previous output value.

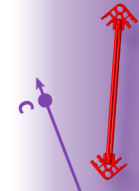
$90^\circ < \angle A < 180^\circ$



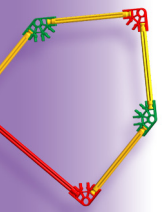
$v = v$
 \overline{AB}



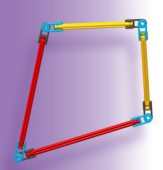
$r + b = b + r$



\overline{CD}



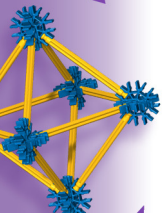
\overline{EF}



$y + s = s + y$



$w < r$
 $y + y + w + w = 2y + 2w$



$90^\circ < \angle A < 180^\circ$

LESSON 16

5. Are all of these ratios equal?

6. Can we use an exponential function to model the number of small triangles y in Sierpinski's triangle at iteration x ?

An exponential function has the form $y = a \cdot b^x$, where a is the initial value of the function, the output value that corresponds to an x -value of 0 and b is the common ratio, the ratio of consecutive output values that are all equal. We also call b the growth factor, if the output values are increasing in value, or decay factor, if the output values are decreasing in value.

7. Look at the table. What is the value of a ?

$a =$

8. Look at the table above. What is the common ratio? This is the value of b .

$b =$

9. Is b a growth factor or a decay factor?

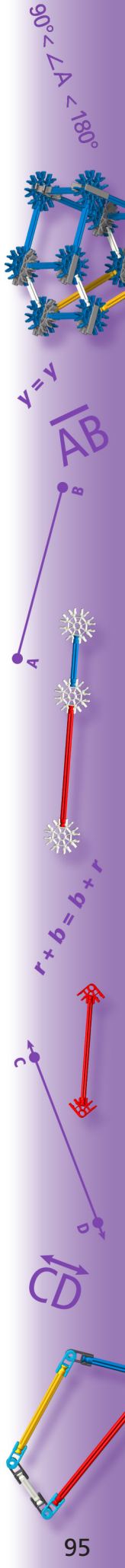
10. Write an equation for our exponential model for the number of small triangles y as a function of the iteration number x .

Exponential Function Model: $y =$

11. Use this model and a calculator to find the number of small triangles in iteration number 5 of Sierpinski's triangle.

12. Plot the five points (iteration #, number of small triangles) from the table (previous page) on a separate sheet of graph paper and connect them with a smooth curve. Because this curve looks like a capital J, we call this curve a "J" curve. The graph shows exponential growth.

13. How could you use the graph of our model above to predict how many iterations it would take to obtain 243 small triangles?



Equivalent fractions: two fractions with the same value; the cross products will be equal.

Exponential decay: an exponential function $y = a \cdot b^x$ with $0 < b < 1$.

Exponential function: a function that can be written in the form $y = a \cdot b^x$, where a and b are real constants.

Exponential growth: an exponential function $y = a \cdot b^x$ with $b > 1$.

Factor: a number or variable multiplied by a monomial to obtain a monomial.

Fibonacci number: the sequence 1, 1, 2, 3, 5, 8, ...

Fractal: a shape that is self-similar; that is, a shape where a part of the shape looks like the whole shape.

Fraction in simplest form: when its numerator and denominator do not have a common factor.

Geometric sequence: a sequence where the quotient of any two consecutive terms is always the same number, called the common ratio.

Golden ratio: the limit of the sequence of ratios of consecutive Fibonacci numbers.

Improper fraction: a fraction with the property that the numerator is greater than or equal to the denominator; can be rewritten as a whole number or mixed number.

Initial value: the data value when the input $x = 0$.

Integers: numbers that consist of a sign and a whole number; the set. $\{\dots, -2, -1, 0, 1, 2, \dots\}$

Iteration: one step in a process where you repeat the same procedure over and over again.

Like terms: two monomials with the same variable part.

Linear function: a function whose graph is a straight line; a function that can be written in the form $y = mx + b$.

Linear polynomial: a polynomial of the form $ax + b$ where a and b are real numbers and a is nonzero.

Mixed number: a sum of a whole number and a fraction.

Monomial: a real number or the product of a real number and one or more variables.

Multiplicative identity: the number 1 has the property that any number times 1 equals 1 times that number which equals that number. In symbols, $1 \cdot a = a \cdot 1 = a$.

Numerical coefficient: a number factor; the number multiplied by the variable part of an algebraic term.

Oblique prism: a prism where lateral faces are parallelograms.

Perfect number: a counting number with the property that the sum of its proper factors is equal to the number.

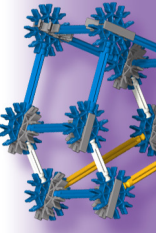
Platonic solid: a regular polyhedron; closed surface that is convex, the faces are all congruent regular polygons, and every interior dihedral angle is congruent.

Polyhedron: a closed surface composed of polygons that each lie in a different plane with the property that the polygons meet at line segments called **edges** with only two polygons meeting at each edge. The corners of a polyhedron are called **vertices**. The polygons that form the surface are called **faces** of the polyhedron.

Polynomial: the sum of monomials.

Prime: a counting number with exactly two counting number factors.

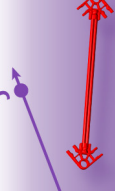
$90^\circ < \angle A < 180^\circ$



$y = y$



$r + b = b + r$



CD



Prism: a polyhedron formed of two congruent polygons in parallel planes (the **bases**) and side faces that are parallelograms.

Product: the answer to a multiplication of two numbers.

Proper factor: a counting number factor of a counting number c that is less than the number c .

Pyramid: a polyhedron formed of a polygon (the **base**) and triangular side faces that meet at a single point off the plane containing the base called the **apex**.

Quadratic polynomial: a polynomial of the form $ax^2 + bx + c$ where a , b , and c are real numbers and a is nonzero.

Quotient: the answer to a division of two numbers.

Recursive formula: a formula that uses one or more previous values of a sequence to obtain the next value in the sequence.

Reflexive property: a number or number expression is equal to itself. In symbols, $a = a$.

Sequence: a set of numbers separated by commas.

Similar polygons: polygons with the properties that 1) the corresponding interior angles have the same measure and 2) the ratios of corresponding sides equal a constant.

Similar polyhedra: polyhedra that 1) have the same number of faces, 2) have corresponding faces that are similar polygons, 3) have the ratios of corresponding edges that are all the same, and 4) have all corresponding dihedral angles are congruent.

Simple closed surface: a surface that divides space into three distinct regions: 1) points in the interior of the surface, 2) points on the surface, and 3) points in the exterior of the surface.

Slope: the steepness of a line; the ratio of the change in the y -values to the change in the x -values of any two points on a line.

Slope-intercept form of a line: $y = mx + b$, where m is the slope of the line and b is the y -value of the y -intercept of the line.

Surface area of a polyhedron: the sum of the areas of the faces of the polyhedron.

Symmetric property: if one number or number expression is equal to a second number or number expression, then the second number or number expression is equal to the first number or number expression. In symbols, if $a = b$, then $b = a$.

Term: in a sequence, one of the numbers separated by commas; in a polynomial, an expression separated by a plus or minus sign that includes the sign.

Transitive property: if one number or number expression is equal to a second number or number expression and the second number or expression was equal to a third number or expression, then the first number or expression will equal the third number or expression. In symbols, if $a = b$ and $b = c$, then $a = c$.

Unit: the number 1; a number with exactly 1 factor.

Variable: a quantity that can change or vary; often represented by a letter.

Vector: a line segment with a direction.

Volume: the total amount of space inside a closed three-dimensional shape.

y -intercept: the point where a graph intersects the y -axis; the y -value when $x = 0$.

